



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

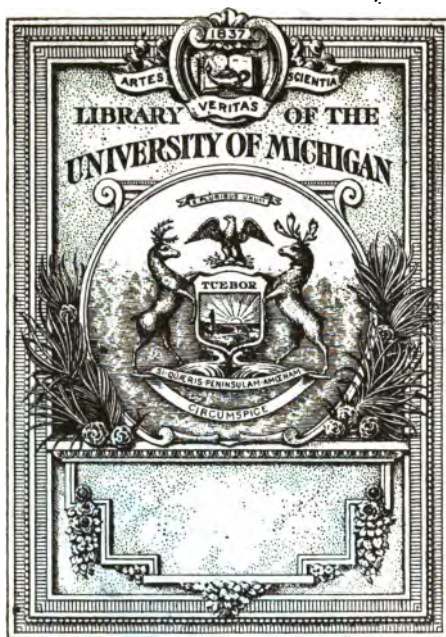
We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>

Wakson



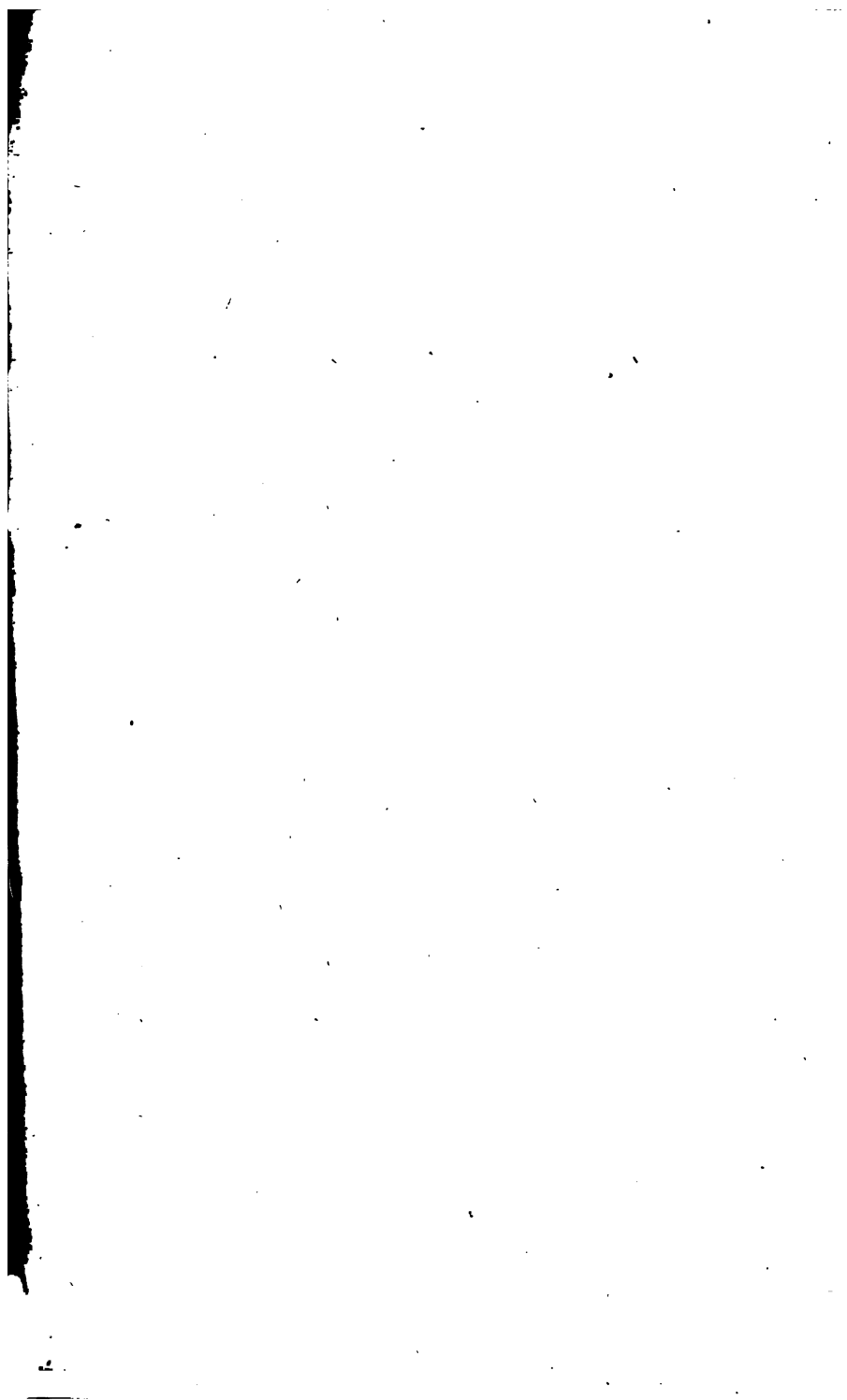
be
1

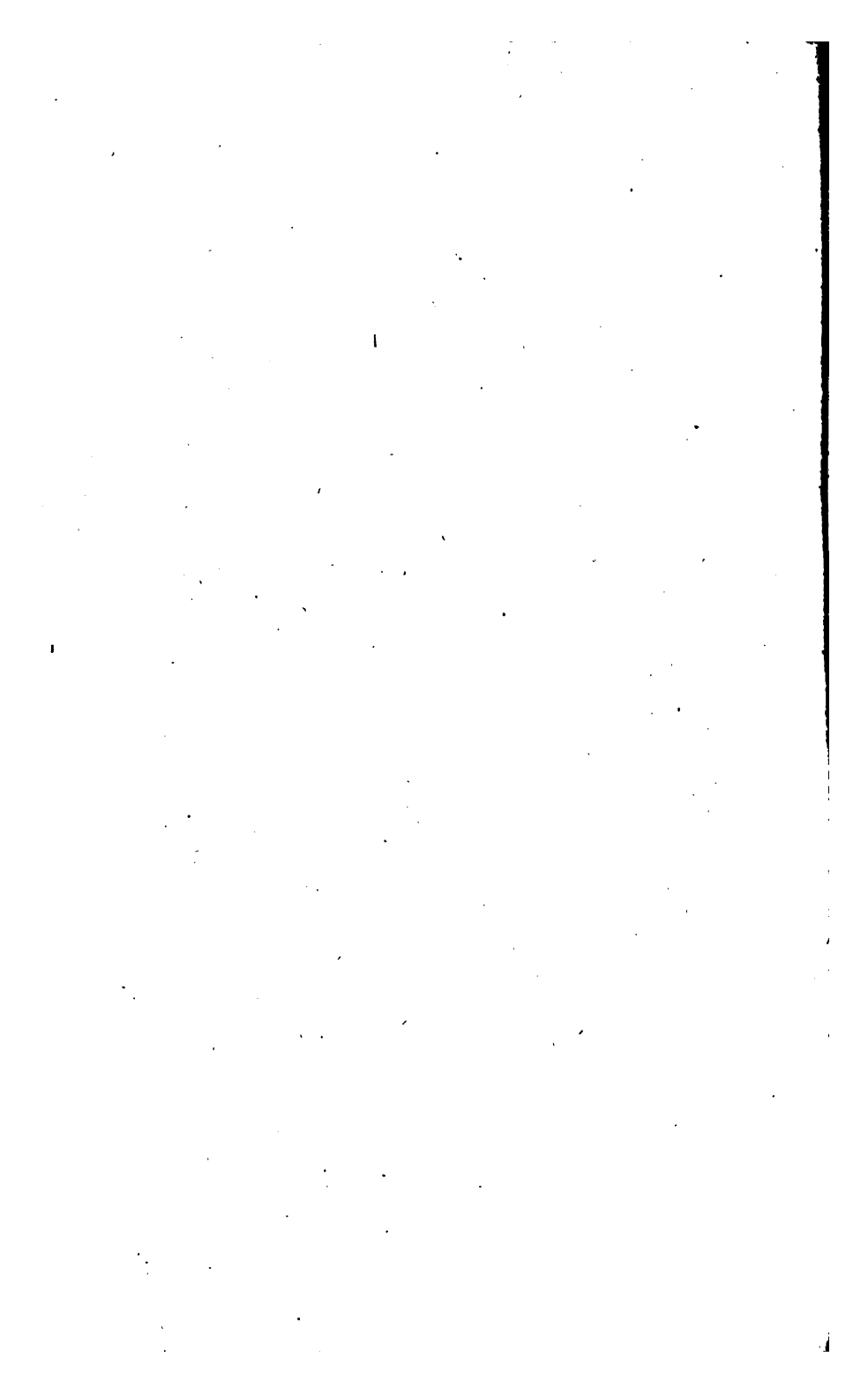


The 1st Book contains 28 Propositions
2^d _____ 27 _____
3 _____ 39 _____

QA
35
J12 *al*







ELEMENTS

OF

CONIC SECTIONS

IN THREE BOOKS:

In which are demonstrated the principal
Properties of the PARABOLA, EL-
LIPSE, and HYPERBOLA.

By RICHARD JACK,
Teacher of Mathematicks in *Edinburgh*.



EDINBURGH,

Printed by THO. WAL. and THO. RUDDIMANS.
M. DCC. XLII.

Entred in *STATIONERS-HALL*,
according to Act of Parliament.



T O

The Right Honourable,

J O H N

Lord *H O P E*,

My LORD,

THE great perfection your Lordship has arrived at in the knowledge of Arts and Sciences, makes me presume to dedicate to your Lordship the following sheets. Should I follow the common practice of Authors, it would offend that native modesty which renders

a 2

your

D E D I C A T I O N.

your Lordship no less conspicuous, than
your regard for, and inclination to pro-
mote Learning ; I shall therefore forbear,
and only beg leave to conclude, that I am,
with the greatest respect,

My LORD,

Your Lordship's

most obedient humble Servant,

RICH^d. JACK,



P R E F A C E.

MATHEMATICKS, which is of the utmost importance with respect to our commerce, security, and, in general, all the affairs of life, has, for the most part in all its branches, been, by very able mathematicians, treated of in the English language in a full and learned manner. That which is most defective seems to be the Conic Sections, few having, in all their mathematical writings, so much as touched on their first principles; and those who have, treat them in such an obscure and intricate manner, as render them intelligible only to those who have made a considerable progress both in the elements of geometry and algebra; by which means Gentlemen are either totally deprived of that valuable part of geometry, which might be of considerable advantage to them, or laid under an indispensable necessity of learning wholly the elements of algebra. There are, it is true, exclusive of Apollonius and the com-
men-

mentators on him, several very learned performances on that subject, by Mydergius, Vincentio, Vivani, De Wit's first book, and De la Hire, whose demonstrations are purely geometrical; but as these learned Gentlemen have, for the general benefit of Europe, wrote in Latin, a performance of this kind seem'd much to be wanted, as there are a great many in Britain who take pleasure in, and have also a genius extremely well calculated for improving mathematical learning, but who have not made such advances in that language, as to render such performances useful to them. And what makes them less serviceable to those who understand the language is, that their demonstrations depend upon a vast field of principles, which must necessarily consume a great deal of time to be tolerably acquainted with; the first three, as well as the ancients, including a great number of lemmas, and their principal properties all demonstrated from the various sections of the cone. Though De Witt and De la Hire, from the example of Dr. Wallis, seem to unvail the writings of the ancients in one particular, by describing the curves on a plane, and from thence demonstrate their principal properties; yet neither of their treatises appear to be rightly calculated for a young geometrician, both being too,
super-

*superficial, especially the first, which contains no more than eighteen propositions; and is very perplexed with respect to their description, from the various intersections of lines cutting one another at certain angles **,

THERE is one not yet mentioned whose performance is worthy of the author, being wrote in such an elegant manner, and with so true a geometrical spirit, as renders it inferior to none of the ancients, and superior to all the moderns. I suppose few will be at a loss to apprehend that I mean Mr. Robert Simson professor of mathematicks in the college of Glasgow. Had that incomparable geometrician been pleased to write in English, though it would have been a considerable loss to Europe, by confining such a valuable work to the scanty limits of Britain, and had he enunciate all his propositions in general, and less compounded his demonstrations, we should have had so complete a system of conic sections, as would have put a stop to writing on that subject a considerable time; but his demonstrations, though worthy
of

* De la Hire published a very full treatise of Conic Sections at Paris 1685, but it labours under the same inconveniencies with those already mentioned.

iv P R E F A C E.

of a great geometrician, are so much compounded, as render them scarcely intelligible to those who have read nothing on the same subject before, at least, it requires a great deal of labour to comprehend them; and the demonstration of a great many of his problems depend upon Euclid's data, which few young geometricians can be supposed to be acquainted with. These considerations chiefly induced me to publish the following treatise, which may be easily read with no more than a tolerable knowledge of the first six books of Euclid, and the first sixteen propositions of the eleventh book.

I shall now proceed to give a sketch of the work itself:

THE generation of the curves I have taken from De l'Hospital, thinking his manner more simple than that of Dr. Wallis, De Witt, De la Hire, or Oznam, and have all along been particularly careful to express the enunciations in general, and in such terms as contain no ambiguity, because it is the enunciation only can communicate to the reader a distinct Idea of the proposition, and by expressing them in general, he is less liable to misapprehend the meaning of a proposition than by particular enunciations;

P R E F A C E. †

ous: For if we consider the unripened ideas of those who are learning the first rudiments of geometry, which may be so confused, that they will only be able to consider the terms made use of to communicate the proposition to them, we shall find this method very necessary. But to make this point still clearer, I shall suppose the sixteenth proposition of the first book of Euclid was stated thus, If the triangle ABC has the side BC produced to D, the angle ACD is greater than the angle ABC or EAB; by expressing it in this manner they may be so far from comprehending the true meaning of the proposition, as to conclude with themselves that this is not an universal property of the whole genus of right lined triangles, but of that particular one; whereas this inconvenience may be easily provided against by enunciating it in general thus, If any right lined triangle has any one of its sides produced, the outward angle is greater than either of the inward opposite angles; where the words made use of are not only universal, but so striking, that the weakest capacity at first must understand the true meaning of it.

THIS Euclid has carefully avoided, being too great a geometrician not to foresee the bad consequences that particular enunciations may

vi P R E F A C E

be attended with, for which he has justly met with the general approbation of the greatest and wisest men that have appeared since his time, which is above two thousand Years. From his example it is, that I have for the most part repeated the enunciations at the end of each proposition, except in the second and third books, where their prolixity would have swelled the book to too great a length; and in the demonstrations I have endeavoured to keep close to his stile; and to render them still more easy, in the most simple part of the constructions, have quoted his authority.

SEVERAL of the moderns of no small note, in their geometrical writings, exclaim against the demonstrations of the ancients, which, from the infancy of geometry until Des Cartes, were justly admired and universally applauded, and in their place substitute arithmetical calculations, which, in their ultimate conclusions, to me appear no more than begging the question, and taking the whole elements of their geometry for granted. For to maintain that these demonstrations, founded upon the calculations of algebraic equations, are just, it is the indispensable business of the demonstrator to demonstrate these principles upon which he builds his demon-

mon-

monstrations, which is a task that no author on algebra as yet, that I know of, has been able to accomplish: But admitting their principles to be really demonstrated, in that case it must still be allowed, that geometrical demonstration is vastly more natural, so far as it is more natural to represent a triangle by a triangle, a circle by a circle, or any other geometrical figure by itself, than, in a dark and obscure manner, by the alphabetical letters a, b, c, &c.

THE advantages the one has above the other cannot be inferior to that which the Egyptians in a superlative degree experienced from the invention of writing, compared with that imperfect method they had before of communicating to one another the ideas they had formed of men and things by hieroglyphicks,

BESIDES, by the method of the ancients, a whole field of beautiful truths presents itself in the very construction of the figure, and various, though equally just, demonstrations of the same or different properties receive their birth in our ideas from the simple comparison of the different parts of the construction; whereas the demonstrations introduced by the arithmetical geometers, from the different steps

b 2 of

of an equation, are not only dark in themselves, but, when they have arrived at their last conclusion, few are capable of forming a just idea of the reasons for taking these steps in that certain and particular order they have made use of to acquire the truth demonstrated: Nay the seventh, eighth and ninth books of Euclid, to me are a pregnant proof, that arithmetical or algebraical calculations, are so far from being proper to demonstrate geometrical theorems, that they are scarce capable of demonstrating their own principles, but must have recourse to geometry; it is therefore surprising that such a learned and accomplished mathematician as Dr. Wallis should express himself in the manner he does, in his opera mathematica, vol. 1. page 296. in his preface to his Conic Sections; he says, * That he rather chooses to demonstrate his propositions by arithmetical calculations, than by lineal demonstrations, as they are no less scientific, and more perspicuous, simple and universal; neither has any one (that he knew) denied

* Demonstrationibus etiam potius ex arithmetico calculo petitis rem agens, quam linearibus; cum illæ nec minus scientificæ sint, & magis perspicuæ, sed & simpliciores sunt atque magis universales; nec quisquam est (quod sciam) qui demonstrationes arithmeticas in geometria admittendas negaverit.

P R E F A C E. ix

denied to admit arithmetical demonstrations into geometry. *And Mr. Jones, in his preface to his synopsis palmariorum matheseos, not only is of the same opinion, but makes use of the very same words with Dr. Wallis. To them I might add several others, which I purposely omit, lest I should swell this preface to a greater length than I intended.*

*I hope, from what I have said, none will suspect me an enemy to algebra; that was by no means my design, but only to show the inconsistency of demonstrating geometrical theorems by it: For my own part, I look upon algebra as one of the valuable and useful parts of the mathematicks, its conclusions are just, and operations universal, and never seems to act out of its sphere, but when applied to geometrical demonstrations. In this Mr. Oznam agrees with me; for though, in his preface to his treatise of Conic Sections, he says, * That that treatise was composed principally for the benefit of those who have an inclination to know the solution of equations of more than two dimensions*

* Ce traité ayant été composé principalement en faveur de ceux qui desireroient savoir résoudre les équations de plus de deux dimensions par le moyen des sections coniques, &c.

fions by the assistance of Conic Sections; yet he for the most part demonstrates his theorems geometrically. But to return:

THE order of the propositions is different from Mr. Simson's, some of his corollaries, which appeared intricate, are made propositions, and some of his propositions are made corollaries; several of his demonstrations are retained, those passages in them which are perplexed being explained.

THERE are some propositions inserted that I found not in any author that I met with, particularly the nineteenth proposition of the first book, with its two first corollaries, the twenty sixth of the second, and thirty second of the third book. The demonstration of the nineteenth proposition of the first book is a little tedious and particular, because the diameter cut by the tangent is on the same side of the axis with the point of contact; and were they on different sides, the same construction and demonstration would not answer; but the general demonstration may be easily conceived from the following construction: From the vertex of the diameter cut by the tangent, draw an ordinate to the diameter which is drawn through the

the point of contact, then will the square of the segment of the tangent, intercepted between the point of contact and the diameter cut by it, be equal to the square of that semi-ordinate that is (*by cor. 2. prop. 13 and prop. 15.*) four times the rectangle contained between the absciss of the diameter (*cut by the tangent*) to the ordinate drawn through the point of contact, and the segment of the same diameter intercepted between the directrix and perpendicular. *The other part of the demonstration goes on in the same manner with that of the proposition itself; the only reason why I made choice of the other was, that from this could not be drawn the two first corollaries, and for the sake of them it was introduced. The twenty second and twenty fifth propositions in the second book may be constructed and demonstrated in the same manner, and respectively in the same words with the thirty fourth and thirty eight propositions in the third book. I designed to add the first principles of the cycloid and cissoid, but the book having already swelled beyond what I first designed, shall leave them to another opportunity.*

ERRATA.

PAge 7. line 20. for less read greater. P. 23. first citation. r. 4 E. l. 1. P. 25. l. 11. r. GE. P. 50. l. 25. for segment r. square. Ibid. l. 26. for square r. segment. P. 55. l. 25. for B r. D. P. 74. l. 18. r. passes through. P. 90. l. 2. r. FE. Ibid. l. 3. for E r. G. P. 111. l. 5. for ellipse r. circle. P. 117. last citation, r. pr. 1. P. 123. l. 17. for MK r. DK. P. 148 and 149. for D r. always B. P. 151. l. 20. for KH r. GH. P. 155. l. 25. r. NOYI. P. 156. r. always ONIY. P. 176. l. 5. r. QP. Ibid. l. 23. r. PR. P. 177. l. 5. dele is. P. 221. l. 9. for BN r. FN. P. 222. l. 14. dele as B. P. 256. l. 4. r. CE. P. 264. l. 18. r. line. Ibid. l. 27. r. square. P. 268. l. 23. for parallel to the second diameter, which is conjugate to CB, r. parallel to the ordinate of the diameter drawn through the point of contact. P. 288. l. 17. r. GD. P. 289. l. 16. r. CB.

page 251 line 2^d. for F & G read A and M.
 page 272 line 2^d. read join EL & BC
 page 297 line 23^d. for squares of AC read square of AC
 page 319 line 19th for Q and B read Q and B.
 page 319 line 18th add join OB.
 page 324 ^{third} citation read 37 of this
 page 324 line 25th read semiminute. **E L E.**

ELEMENTS OF Conic Sections.

BOOK I. OF the *PARABOLA*.

DEFINITIONS.

I. **L**ET *AB* be any right line, and *C* any point taken out of it, and *DEF* a square, which let be placed in the same plane in which the right line and point are, in such a manner that one side of it, as *DE*, be applied to the right line *AB*, and the other side *EF* coincide with the point *C*, and at *F*, the extremity of the side *EF*, let be fixed one end of the thread *FGC*, whose length is equal to *EF*, and the other extremity of it at the point *C*, and let part of the thread, as *FG*, be brought close to the side *EF* by a small pin *G*; then let the

A square

square DEF be mov'd from B towards A, so that all the while its side DE be applied close to the line BA, and in the mean time the thread being extended will always be applied to the side EF, being stop't from going from it by means of the small pin; and by the motion of the small pin G there will be described a certain curve, which is called a semi-parabola.

And if the square be brought to its first given position, and in the same manner be mov'd along the line AB from B towards H, the other semi-parabola will be described.

This curve line may also be extended from the point C to a distance greater than any given distance; viz. If a square be taken, the length of whose side EF exceeds that given distance, also the whole curve line will lie between the line AB and the point C.

II. *The right line AB is called the line of direction, or the directrix.*

III. *The point C is called the focus of the parabola.*

IV. *All*

IV. *All right lines, that are drawn at right angles to the line of direction, are called diameters; and these points where they cut the parabola are called the vertexes of these diameters.*

V. *The diameter which passes through the focus is called the axis of the parabola, and its vertex is called the principal vertex.*

VI. *A right line which is terminated both ways by the parabola, and bisected by any diameter, that line is called an ordinate to the diameter.*

VII. *That part of any diameter which is intercepted between an ordinate and the vertex of that diameter is called the absciss of that ordinate.*

VIII. *A right line, which is four times that segment of the diameter which is intercepted between the line of direction and its vertex, is called the parameter or latus rectum of that diameter.*

IX. *A right line which touches the parabola in one point, and being produced both ways falls without it, is called a tangent to the parabola.*

PROPOSITION I.

THEOREM I.

If from any point in a parabola a right line be drawn at right angles to the line of direction, that line will be equal to a right line drawn from the same point to the focus,

LET GIK be any parabola, whose directrix is AB, and focus C, and from any point in it, as G, the right line GE is drawn perpendicular to the directrix AB, cutting it in E; and let the line GC be drawn from the same point to the focus. I say, GE is equal to GC.

For let EF be equal to the length of the side of the square, with which the parabola was described, then, by the generation of the parabola, the line EF is equal to the generating thread FGC; and if from both

Book I. *Elements of Conic Sections.*

both the common part FG be taken, there will remain GE equal to GC.

Therefore, *a right line drawn from any point in the periphery of the parabola, perpendicular to the directrix, is equal to a right line drawn from the same point to the focus, which was to be demonstrated.*

Coroll. From hence it follows, that the segment of the axis, which is intercepted between the directrix and the focus, is bisected in the principal vertex of the parabola.

PROPOSITION II.

THEOREM II.

If the distance of any point from the focus of a parabola be equal to a line drawn from the same point perpendicular to the line of direction, that point will be in the periphery of the parabola.

LET there be any parabola, whose line of direction is AB, and its Focus C, and D any point taken in such a manner,
that

that DC its distance from the focus is equal to DE, a right line drawn from the same point perpendicular to the directrix AB. I say, that D will be a point in the periphery of the parabola.

* *Defn. 1.* For if D is not in the periphery, * then must the parabola cut the lines DC or DE in some other point. First, if possible, let it

† *12 E. l. 1.* cut the line DC in F, and from F † draw FG perpendicular to AB, and join EC. Then, because F is a point in the periphery of

‡ *Prop. 1.* the parabola, FG is equal to FC; † also, of *this.* because DE is equal to DC, the Angle DEC is equal to the angle DCE. But the angle DEC is equal to the angle FHC,

* *29 E. l. 1.* * therefore the angle FHC is equal to the angle FCH, and consequently the side FH

† *6 E. l. 1.* is equal to the side FC †. But FC is equal to FG, therefore FH is equal to FG, a part to the whole, which is absurd.

Neither can the parabola cut the line DE in any other point than D. For, if possible, let it cut it in I, then will IC be † *1 of this.* equal to IE. † Add ID to both, so that DI and IC will be equal to DE. But DE, by the hypothesis, is equal to DC, therefore DC is equal to DI and IC, that is,

one

one side of a triangle equal to the other two sides, * which is absurd.

* 20 E. I. 1.

Therefore, if the distance of any point from the focus of a parabola, be equal to a line drawn from the same point perpendicular to the line of direction, that point is in the periphery of the parabola, which was to be demonstrated

Cor. 1. From hence it follows, that the distance of any point from the focus, which is between the periphery and the axis, is less than a right line drawn from the same point perpendicular to the line of direction; because, by the demonstration, FH was proven equal to FC. But FH is less than FG, therefore FC is also less than FG.

Cor. 2. Hence it also follows, that the distance of any point without the parabola, from the focus, is ^{*} less than a line drawn perpendicular from the same point to the line of direction. For, because the angle DCE is equal to the angle DEC, therefore the angle ICE is less than the angle IEC; and consequently

* greater

* the side IC is greater than the side IE. * 19 E. I. 2.

Cor.

¹ Nam in Cor. 3. From hence it likewise follows,
 primo casu non that if the distance of any point from
 notent esse punctum the focus of a parabola be either less or
~~peripheriam~~ illud greater than a line drawn from the same
 extra parabolam point perpendicular to the directrix, in
 nec in secundo the first case the point is either between
 intra, per the periphery of the parabola and its fo-
 hanc prop. In cus, or in the axis; and in the last case
 neutro autem the point is without the circumference.
 notent esse in
 ipsa parabola
 per nom. hij.

PROPOSITION III.

THEOREM III.

If any right line be drawn through the focus, that line produced will cut the periphery of the parabola.

LET AB be any parabola, and its focus C, and through C let any right line, such as CD, be drawn. I say, that CD produced will cut the parabola.

For, produce the axis CB, and let it cut the directrix in E. Bisect the angle ECD

*9 E. I. I. * with the line CF, which cuts the directrix
 †31 E. I. I. in F, and through F, draw FG † parallel to
 the axis EC. Then, because the line FG
 is parallel to EC, the angle ECF is equal
 to

to the angle CFG: † But the angle ECF is †^{29 E. I. 1.} equal to the angle FCD, therefore the angle CFG is equal to the angle FCD; and because the angle ECD is less than two right angles, therefore the two angles CFG, FCD taken together are less than two right angles, and consequently the two lines CD and FG produced will meet, which let be in the point H; that point H is a point in the periphery of the parabola. For, because the angle HCF is equal to the angle HFC, the right line HC is equal to the right line HF*, and consequently *^{6 E. I. 1.} the point H is a point in the periphery of the parabola. † †^{2 of this.}

Therefore, *if any right line be drawn through the focus, that right line produced will cut the periphery of the parabola, which was to be demonstrated.*

PROPOSITION IV.

THEOREM IV.

Any right line drawn perpendicular to the line of direction cuts the parabola in one point only, and below that point the line will fall within the parabola.

B

LET

LET AB be any parabola, its focus C, and line of direction DE; and let DA be drawn perpendicular to the directrix DE. I say, DA will cut the parabola in one point only, and below the point A will fall within the parabola.

For, join DC, and with the line DC, and at the point C, construct the angle ^{* 23 E. l. 1.} DCA equal to the angle ADC *, and let the line AC meet DA in A. Then, because the angle ACD is equal to the angle ^{† 6 E. l. 1.} ADC, the line AC is equal to AD, † and ^{‡ 2 of this.} consequently the point A ‡ is a point in the periphery of the parabola; and if any point be taken in the line DA produced, as F, I say, that point F is within the parabola. For, join FC; then, because the angle ACD is equal to the angle ADC, therefore the angle FCD is greater than the angle FDC, and consequently the side FD ^{* 19 E. l. 1.} opposite to the greater angle FCD is * greater than the side FC opposite to the smaller ^{† Cor. 1. pr. 2 of this.} angle; therefore the point F † is within the parabola.

Therefore, *any right line drawn perpendicular to the line of direction cuts the parabola in one point only, and being*

ing produced, will always fall within the parabola; which was to be demonstrated.

PROPOSITION V.

THEOREM V.

If a right line be drawn from any point in the periphery of a parabola, bisecting an angle which is contain'd between two right lines drawn from the same point, viz. one to the focus, and the other perpendicular to the directrix, that line is a tangent to the parabola. Also, a right line drawn perpendicular to the axis, from the principal vertex, is a tangent to the parabola.

FIRST, let A be any point taken in the periphery of the parabola AB, whose focus is C, and directrix DE; and let AD be drawn perpendicular to the directrix DE, and AC join'd to the focus, and let AF be drawn bisecting the angle DAC. I say, the line AF is a tangent to the parabola.

For, in the line AF produced take any
B 2
point,

12 *Elements of Conic Sections. Book I.*

point, such as G, and join GD, GC and DC, and from G draw GH perpendicular * to the directrix, and cutting it in H. Then, because A is a point in the parabola, AD is equal to AC, † and AF common, and the angle DAF equal to the angle CAF; therefore DF is ‡ equal to FC, and the angle AFD equal to the angle AFC. Again, because DF is equal to FC, and FG is common, and DFG equal to the angle GFC, therefore GD is equal to GC. But because the triangle GHD is right angled at H, GD is * 19 E. 1. 1. greater than GH, * wherefore GC is also † Cor. 3. pr. greater than GH; therefore the point G † 3 of this. is without the parabola; and after the same manner may any other point in the line AF be proved to be without the parabola; † Def. 9 of this. consequently the line AF † is a tangent to the parabola. W. W. D.

Secondly, let the line BI be drawn from B the principal vertex perpendicular to the axis BC, I say, BI is a tangent to the parabola. For, let any point be taken in the line BI, such as K, and from K draw KL perpendicular to the directrix DE, which cuts it in L, and join KC to the focus.

Then,

Then, because the angle CBK is a right angle, KC * is greater than CB; but CB * 19 E. 1. 1, is † equal to BE, and KL is equal to BE, † Cor. pr. 1 of this, † therefore KL is equal to BC, and consequently KC is greater than KL, therefore the point K * is without the parabola. * Cor. 3. pr. 2 of this. And after the same manner may it be demonstrated that any other point in the line BI will be without the parabola, therefore the line BI † is a tangent to the parabola. † Def. 9 of this.

Therefore, *if a right line be drawn from any point in the periphery of the parabola bisecting an angle which is contained between two right lines drawn from the same point, the one to the focus, and the other perpendicular to the directrix, that line is a tangent to the parabola. Also, a right line drawn perpendicular to the axis from the principal vertex, is a tangent to the parabola; which was to be demonstrated.*

Cor. From the above proposition it is evident how from a given point in the periphery to draw a line which will be a tangent to a parabola, if the focus and line of direction be given in position.

PRQ.

PROPOSITION VI.

PROBLEM I.

The position of the focus and directrix of any parabola, and a line which is not parallel to any of its diameters, being given, to draw a tangent to the parabola parallel to this given line.

LET AB be the directrix, and C the focus of any parabola, and DE the given right line, which is not parallel to any diameter. It is required to draw a tangent to the parabola which will be parallel to DE.

_{12 E. l. 1.} Through the focus C draw CF perpendicular to DE, which produced cuts the
 †_{10 E. l. 1.} directrix in A, and bisect AC † in G, and
 ‡_{31 E. l. 1.} through G draw GH parallel ‡ to DE. I say, GH is a tangent to the parabola, and parallel to the given line DE. For, thro' the point A draw a diameter to the parabola, cutting the line GH in H, and join HC. Then, because the Angle CFE is a right angle, and the line GH is parallel to the line DE, therefore the angle AGH is
 _{29 E. l. 1.} a right angle; and because AG is equal
 to

to GC, and GH common, and the angle AGH equal to the angle CGH, therefore AH is equal to HC, and the angle AHG equal to the angle CHG; therefore the point H is a point † in the periphery of † 2 of this. the parabola, and the line GH † a tangent † 5 of this. to the parabola; consequently the line GH is drawn a tangent to the parabola, and parallel to the given line DE; *which was to be done.*

Cor. 1. Hence it follows, that any tangent will bisect that line which is drawn from the focus to that point of the directrix where a diameter drawn from the point of contact cuts it.

Cor. 2. From hence it also follows, that from any point in the periphery of a parabola no more than one tangent to the parabola can be drawn; for if it were possible to draw any more, in that case the line intercepted between the focus and that point of the directrix where a diameter drawn from the point of contact cuts it, will be bisected by two or more different lines cutting it in different points, which is absurd.

PRO-

PROPOSITION VII.

THEOREM VI.

An angle, which is comprehended between a diameter and a right line drawn from its vertex to the focus, will be bisected by a tangent drawn from the vertex of the same diameter.

LET AE be a tangent to the parabola AB, touching it in the point A, and from the point of contact A, let AC be drawn to the focus, and AD a diameter cutting the directrix in D. I say, the line AE bisects the angle DAC. For if AE does not bisect the angle DAC, let the line AF bisect the angle DAC. Then, because AF bisects the angle DAC, which is contained between the diameter AD and a line drawn from its vertex A to the focus, consequently AF is a * tangent to the parabola AB, and touches it in the point A. But, by the hypothesis, AE is a tangent to the same parabola, and touches it in the same point A, therefore from the same point A are drawn two lines, AE and AF,

AF, both tangents to the same parabola AB, which † is absurd.

† Cor. pr. 6
of this.

Therefore, *the angle, which is contained between any diameter and a line drawn from its vertex to the focus, is bisected by a tangent drawn from the vertex of the same diameter; which was to be demonstrated.*

Cor. 1. From hence it follows, that if AD be a diameter, and AE a tangent drawn from its vertex, and if the angle EAC be equal to the angle EAD, the line AC will pass through the focus; or if AC pass through the focus, and AE a tangent, and if the angle DAE be equal to the angle EAC, then is the line AD a diameter.

Cor. 2. Hence it also follows, that if from any point in the periphery of a parabola, as A, which is not its principal vertex, a right line, as AE, be drawn a tangent, and AD a diameter, the angle DAE, which is contained between the diameter and the tangent, that lies without the diameter, is less than a right angle, because the angle DAC is less than two rights, and the angle DAE is half
C the

the angle DAC, consequently the angle DAE is less than a right angle.

PROPOSITION VIII.

THEOREM VII.

If from any point in the periphery of a parabola a tangent be drawn, and produced until it cut the axis; and from the same point a line be drawn perpendicular to the axis, that segment of the axis which is intercepted between the tangent and perpendicular line will be bisected in the principal vertex.

LET A be any point taken in the periphery of the parabola AB, whose focus is C and directrix DG, and let AE be a tangent, which produced cuts the axis in E; and from A let AH be drawn perpendicular to the axis. I say, that EH is bisected in the principal vertex B.

For, through A* draw AD perpendicular to the directrix, and join AC to the focus, and let the axis BC cut the directrix in G.

Then, because AH is perpendicular to GH,

GH, and DG is perpendicular to the same line GH, therefore DG and AH are * paral- * 28 E. I. 1.
 lel to one another, and DA is parallel to GH; consequently DH is a right angled parallelogram. Also, because DA is parallel to EH, the angle DAE is † equal to † 29 E. I. 1.
 AEC; but the angle DAE is † equal to the † 7 of this. angle EAC; therefore the angle AEC is equal to the angle EAC, and consequently the side EC is * equal to the side AC; but * 6 E. I. 1.
 DA is † equal to AC; therefore DA is equal to EC, and DA is equal to † GH, † 34 E. I. 1.
 consequently GH is equal to EC; but GB is equal to BC; take BC and BG from both, and there will remain BE equal to BH, consequently EH is bisected in the principal vertex B. .

Therefore, if from any point in the periphery of a parabola a tangent be drawn and produced until it cut the axis, and from the same point a line be drawn perpendicular to the axis, that segment of the axis which is intercepted between the tangent and perpendicular, is bisected in the principal vertex; which was to be demonstrated.

PROPOSITION IX,

THEOREM VIII.

If from any point in the periphery of a parabola a right line be drawn, neither parallel to the axis, nor bisecting the angle which is comprehended between a diameter and a line drawn from the same point to the focus, that line will cut the parabola in one other point only.

LET A be any point taken in the periphery of the parabola AB, whose focus is C, and directrix DE; and from A let the line AF be drawn, neither parallel to the axis, nor bisecting the angle which is comprehended between the diameter and right line drawn from the same point to the focus. I say, that line will cut the parabola in one other point only. This line AF may either pass through the focus or not; and first, let it pass through the focus C; then from the focus C draw a line perpendicular to AF*, meeting the line of direction in the point D; and from the point A draw AG parallel to the axis, which meets the directrix in G, and make
DH

Book I. *Elements of Conic Sections.* 21

DH equal to DG, and through H draw HI parallel to GA, meeting AF produced in I. I say, the point I is a point in the periphery of the parabola. For, join CG and CH.

Then, because the angles DCA and AGD are right angles, and AG* is equal to AC, * 1 of this; the angle ACG is equal to the angle AGC, and consequently the remaining angle DGC is equal to DCG; therefore the side DG† is equal to DC; but DG is equal to DH, † 6 E. 1. 1; therefore DC is equal to DH, and the angle DCH equal to the angle DHC; but the angle DCI is equal to the angle DHI, because both are right angles; therefore the remaining angle ICH is equal to the remaining angle IHC, and the side IC is † equal to the side IH, † 6 E. 1. 1; consequently the point I † is a point in the parabola; there- † 2 of this; fore the line AF cuts the parabola in another point. W. W. D.

Secondly, when the right line AF does not pass through the focus, the construction remaining the same as before, then, about the center A, with the distance AC, describe a circle cutting the line DC again in the point K, and through the points K, C and H describe* the circle KCH. Then, * 3 E. 1. 4
be-

22 *Elements of Conic Sections.* Book I.

because AC is equal to AG, and the angle AGD is a right angle, the line GD is a
 ¶ 16 E. 1. 3. tangent to the circle † GCK, wherefore the
 ‡ 16 E. 1. 3. square of GD ‡ is equal to the rectangle
 contain'd between the lines KD and DC;
 but, by the construction, GD is equal to
 DH, therefore the square upon DH is equal
 to the rectangle contained between
 the two lines KD and DC, wherefore DH
 ¶ 17 E. 1. 3. * is a tangent to the circle HCK; now
 since DH is a tangent, and the angle DHI
 is a right angle, the center of the circle
 ¶ 19 E. 1. 3. HCK * will be in the line HI. Also, be-
 cause the line CK is drawn in the circle
 GCK, and cut by the line AF, which is
 drawn through the circle at right angles;
 ‡ 3 E. 1. 3. therefore the line CK † will be bisected in F,
 wherefore the center of the circle HCF
 will be in the line FI ‡; but before the cen-
 ter was in the line HI, and now in the line
 FI, therefore the center will be in the
 point I, where these two lines cut one an-
 other; and consequently IC is equal to IH,
 ‡ 2 of this. wherefore the point I* is a point in the pe-
 riphery of the parabola. Lastly, I say,
 that the line AF cannot cut the parabola
 in any more points than these two A and I.
 For, if possible, let it cut it in a third
 point

point, as L; join LK and LC. Then, because CF is equal to FK, and FL is common, and the angles CFL and KFL are equal to one another, both being right angles, therefore LC* is equal to LK, and from * 4 E. I. 24 L draw LM perpendicular to the directrix, then is LM† also equal to LC; and so if † 1 of this upon L as a center, with the distance LM, a circle be described, it will pass through the points M, C and K; and because DM is at right angles to DL, DM ‡ will be a ‡ 16 E. I. 3 tangent to the circle MCK, and consequently the square of DM is equal* to the * 36 E. I. 34 rectangle contain'd between KD and DC; but the square of GD is equal to the rectangle contained between KD and DC; consequently the square of GD is equal to the square of MD, and the line GD will be equal to the line MD, a part to the whole, which is absurd. Therefore the line AF cannot cut the parabola in any more points than these two A and I.

Wherefore, *if from any point in the periphery of a parabola a right line be drawn which is neither parallel to the axis, nor bisecting the angle which is comprehended between a diameter and a line drawn from the same point to the focus,*

focus, that line will cut the parabola in one other point only; which was to be demonstrated.

PROPOSITION X.

THEOREM IX.

If to any right line, a line be drawn from the focus of a parabola perpendicular, which produced cuts the directrix; and if the segment of that perpendicular line intercepted between the focus and the given line, be not greater than the segment of the same line, intercepted between the given line and directrix, that given line produced will meet the parabola.

LET AB be the given right line, and C the focus, and from the focus C let CD be drawn perpendicular to AB, which produced meets the directrix in E, and if CD be not greater than DE, I say that AB produced will meet the parabola.

Case 1. If the segment CD be equal to the segment DE, then it is evident that the right line AB will touch the parabola

in

Book I. *Elements of Conic Sections.* 25

in that point where a diameter meets the parabola, which is drawn from the same point E *.

* *cor. pr. 6 of this.*

Case 2. But if CD be less than ED, produce ED, and make DF equal to DC. Then find a mean proportional † between the two lines FE and EC, which let be EG; make EH equal to EG, and about the points F, C and G, describe the circle † GCF, and from G draw GI perpendicular to GE*, which meets the line AB produced in I; then because FE is to EG as EG is to EC, the rectangle contained between FE and EC is equal to the square of EG †; therefore EG is a tangent to † the circle GCF, And consequently the center of the circle will be in the line GI*; then because CF is bisected in D, and from D the line DI is drawn at right angles to it, the center of the circle † will be in the line DI; therefore the Point I will be the center of the circle GCF; and consequently IG is equal to IC; wherefore I is a point † in the parabola. After the same manner † may it be demonstrated, that if from the point H a line be drawn perpendicular to the directrix, and produced meets the line AB, that point where it meets the line AB

D

is

is also a point in the parabola.

Therefore, *If a line be drawn from the focus of a parabola perpendicular to any other line, and produced until it cut the directrix; and if the segment of the perpendicular line, intercepted between the focus and the other line, be not greater than the segment of the same perpendicular, which is intercepted between the other line and directrix, that other line produced will meet the parabola; which was to be demonstrated.*

PROPOSITION XI.

THEOREM X.

If any right line be drawn through a point within a parabola, that line if produced will cut the parabola.

FOR first, if the right line is a diameter it will cut the parabola, and every where beyond that point will fall within it.

Secondly, if the right line is not a diameter, such as AB, which is a right line, drawn any how from the Point A within the Parabola; and let CD be drawn from the

the

the focus perpendicular to AB, meeting the line of direction in E, and join AE and AC: Then because the point A is within the parabola, a line drawn from the point A, perpendicular to the line of direction, will be * greater than AC which ^{* cor . 1. pr. 2 of this.} is drawn from the same point A to the focus; therefore if AE be perpendicular to the line of direction, it will be greater than AC; but if AE be not perpendicular to the directrix, it will be † greater than the † ^{19 E. l. 1.} perpendicular, and therefore much greater than AC: But the square of AE is ‡ equal ‡ ^{47 E. l. 1.} to the two squares described upon AD and DE, and the square of AC is equal to the squares described upon AD and DC; consequently the two squares described upon AD and DE are greater than the sum of the two squares upon AD and DC; take the square of AD from both, there will remain the square of DE, greater than the square of DC; therefore the line DE is greater than DC, consequently the line AB * produced will cut the parabola. ^{* 10 of this.}

Therefore, *If any right line be drawn through a point within the parabola, that line, if produced, will cut the parabola; which was to be demonstrated.*

PROPOSITION XII,

THEOREM XI.

Any right line terminated both ways by a parabola, and parallel to a tangent, will be bisected by a diameter drawn through the point of contact; or that line will be an ordinate to the diameter.

LET there be any right line, such as AB, terminated both ways by the parabola in A and B, and parallel to the tangent EH; through the point of contact E let the diameter EG be drawn, cutting the line AB in G; I say, AG is equal to GB, for AB is an ordinate to the diameter FG.

For let the diameter GE meet the directrix FD in the point F, and from the points A and B let the lines AI and BK be drawn * perpendicular to the directrix, and from the focus C let CF be drawn, meeting AB in L, and about the center A with the distance AI let a circle be described, cutting FC again in the point M, and join CB and BM.

Then because the angle AIF is a right angle,

angle, the line IF is a * tangent to the ^{* cor. 16.} circle ICM, and the angle ENF is a † right ^{E. l. 3.} angle; also because EN is parallel to AB, ^{† cor. pr. 6} of this. the angle ALN is a † right angle; and ^{† 29 E. l. 1.} since the line AB is drawn through the center, cutting the line CM at right angles, CL is * equal to LM. Now because CL ^{* 3 E. l. 3.} is equal to LM, and LB is common, and the angle CLB equal to the angle BLM, both being right angles, therefore the line BC is † equal to the line BM. But BC is ^{† 4 E. l. 1.} † equal to BK; therefore if about the cen- ^{† 1 of this.} ter B, with the distance BK, a circle be described, it will pass through the points K, C and M, and consequently KD will be a * tangent to the circle KCM. Again, ^{* cor. 116. E. l. 3.} because IF is a tangent to the circle ICM, and FM cuts it in the points C and M, the square of IF is † equal to the rectangle ^{† 36 E. l. 3.} contained between MF and FC; and for the same reason the square of FK is equal to the same rectangle contained between MF and FC; therefore the square of IF is equal to the square of FK, and consequently the line IF is equal to the line FK. But because the lines IA, FG and KB, are all parallel to one another, and cut the two lines IK and AB, IF † will be to FK as ^{† 2 E. l. 6.}

AG

AG is to GB; but IF is equal to FK, therefore AG is equal to GB, consequently AB is bisected in G, and AB is an ordinate to the diameter EG.

Therefore, *Any right line terminated both ways by the periphery of a parabola, and parallel to a tangent, is bisected by a diameter drawn through the point of contact, or is an ordinate to that diameter; which was to be demonstrated.*

Cor. 1. From hence, and prop. 5. it follows, that a line drawn from any point in the periphery of a parabola, perpendicular to the axis, is a semi-ordinate to the axis; also an ordinate drawn to the axis through the focus, is equal to the *latus rectum* of the axis.

Cor. 2. From hence it follows, that if any right line, as AB, which is terminated both ways by a parabola, be bisected by any diameter EG, that line will be parallel to the tangent which is drawn through the vertex E of the diameter; for if the tangent drawn through the vertex E be not parallel to AB, let a tangent be drawn * parallel to AB, then will a diameter which passes

* 6 of this.

ses through the point of contact (by the above proposition) bisect the line AB; but the diameter EG by the hypothesis bisects the line AB, therefore the line AB is bisected by different diameters; *which is absurd.*

Cor. 3. All the ordinates to one and the same diameter are parallel to one another.

Cor. 4. Two or more right lines terminated by a parabola, and parallel to one another, that diameter which bisects any one of them will bisect them all; for that one which is bisected is * parallel to a tangent which is drawn † cor. 2 of this pr. from the vertex of that diameter; therefore they are all † parallel to the same tangent, and consequently the diameter drawn through the point of contact will bisect them all. † 30 E. l. 11.

Cor. 5. That right line which bisects two or more right lines, parallel to one another, and terminated both ways by a parabola, is a diameter; for if not that diameter which bisects one of these lines, will by the preceeding corollary bisect them all, and consequently every one of them will be bisected by

by another right line ; *which is absurd.*

* *Contigenti enim
per Verticem sunt
parallelae ut est
indes se omnes
ordinatim applicatae.*

Cor. 6.* A right line which is drawn from the vertex of any diameter, parallel to an ordinate to the same diameter, is a tangent to the parabola.

*Unica autem recta per punctum illud duci potest cui sint
istae parallelae. Ergo hanc unicam contingentem esse
recipere est.*

PROPOSITION XIII.

THEOREM XII.

If from any point in the periphery of a parabola a right line be drawn perpendicular to any diameter, also an ordinate to the same diameter, the square described upon the perpendicular will be equal to the rectangle contained between the absciss and the latus rectum of the axis.

Case 1. **W**HEN the diameter is the axis of the parabola.

Let A be any point taken in the periphery of the parabola AB, whose focus is C, its principal vertex B, and directrix DE ; from A let AF be drawn, cutting the axis BC at right angles, then is AF a semi-

* cor. 1. pr.
12 of this.

* ordinate to the axis ; I say, the square of AF is equal to the rectangle contained between
tween

tween BF and the *latus rectum* of the axis BC, or four times BE : For from A draw AD perpendicular to the directrix, cutting it in D, and join AC to the focus ; Then because the triangle AFC is right angled at F, the square of AC is * equal. * 47 E. 1. 11 to the two squares described upon AF and FC ; but AD is † equal to AC, that is, † 1 of this EF is equal to AC, therefore the square of EF is equal to the sum of the two squares described upon AF and FC ; and because EB is equal to BC, there is a line BF any how cut in C, and to it BE is added equal to BC, therefore ‡ four times the rectangle ‡ 3 E. 1. 24 contained between FB and BC, together with the square described upon CF, is equal to the square of EF ; but the square of EF was before proven equal to the sum of the two squares described upon AF and FC, therefore the two squares described upon AF and FC are equal to four times the rectangle contained between BF and BC, together with the square of FC ; take the square of FC from both, and there will remain the square of AF, equal to four times the rectangle contained between BF and BC ; that is, the square of AF is equal to a rectangle contained between BF,

E

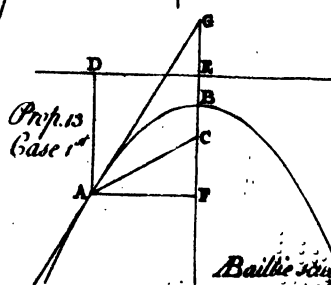
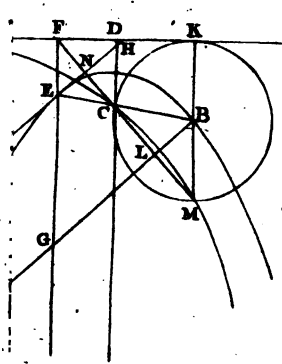
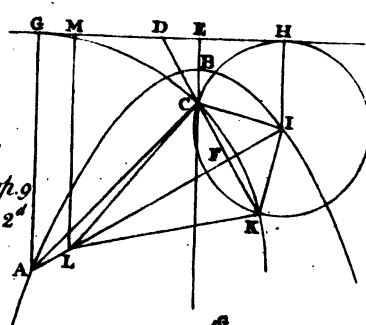
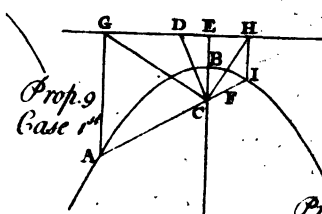
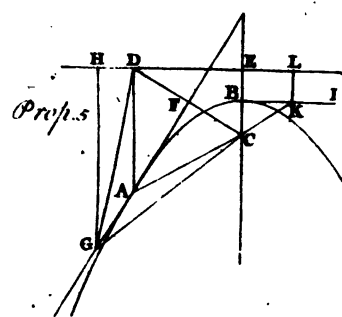
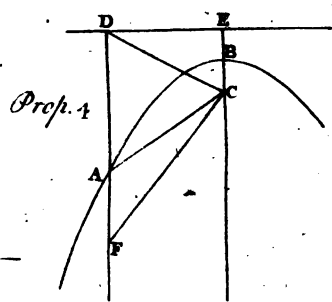
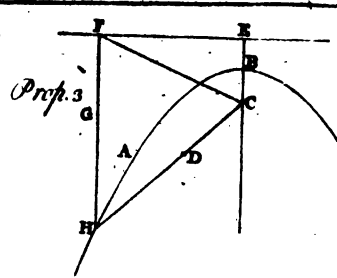
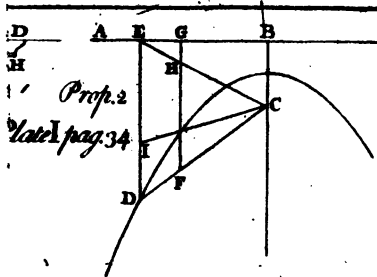
and

and four times BC or BE, which is the
 * def. 8 of * *latus rectum*; therefore the square of
 this the perpendicular or semi-ordinate to the
 axis, is equal to the rectangle contained
 between the absciss and the *latus rectum*
 of the axis; W. W. D.

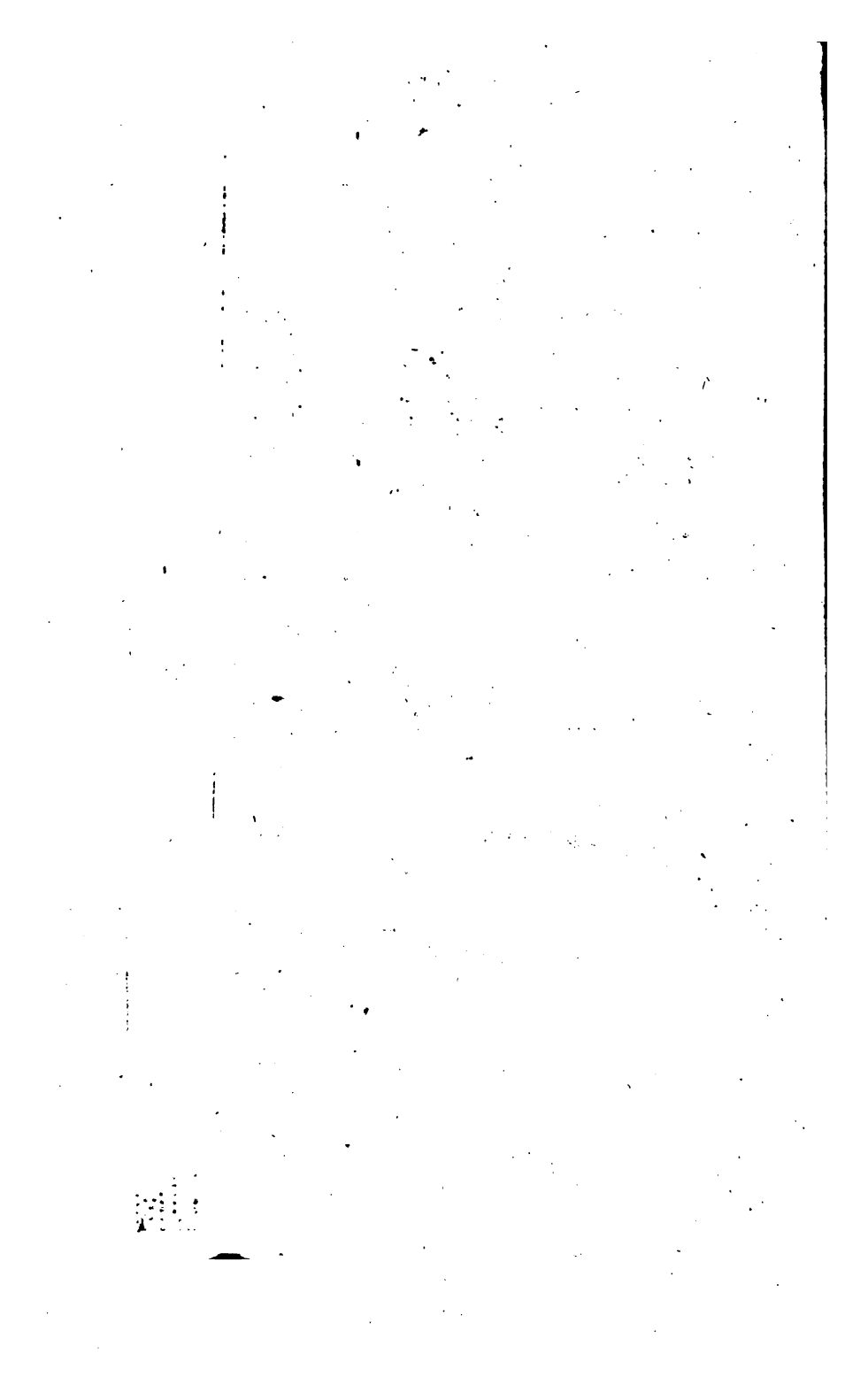
Case 2. When the diameter is not the
 axis.

Let ABD be any parabola whose focus
 is C, its principal vertex B, and directrix
 EF, and from any point in its periphery,
 as A, let AH be drawn perpendicular to
 any diameter, as GH, and from the same
 point A let AK be drawn an ordinate to
 the same diameter, and let G be the ver-
 tex of the diameter; I say, the square
 of AH is equal to the rectangle contained
 between the absciss KG and the *latus re-*
ctum of the axis.

For from G the vertex of the diameter
 † 11 E. 1. 1. draw GL † parallel to AK, meeting the
 ‡ 12 of this. axis in L, which is a ‡ tangent to the pa-
 parabola, and from A draw AM perpendi-
 cular to the directrix; then upon A, as a
 center with the distance AM, describe a
 circle touching the directrix in the point
 M, which will pass through the focus C,
 and join FC, which produced cuts the
 circle



Baillie sculp.



circle again in N, and the lines GL and AK in the points O and P, and let AK produced cut the axis in Q.

Then because the angles COL, CEF, are both right angles, and the angle FCE is common, the two triangles FCE, COL, are * similar; therefore as FC is to * 4 E. 1. 6.
CE, so is CL to CO; but as CL is to CO, so is † LQ to OP: Therefore as † FC is † 2 E. 1. 6.
to CE, so is LQ to OP; consequently the † 11 E. 1. 5.
rectangle contained between FP and OP
is equal to the rectangle contained between CE and LQ. Again, because FC
is * double OC, and CN is double of * cor. pr. of this.
CP, FN is double OP; consequently the rectangle contained between NF and FC is equal to double the rectangle contained between FC and OP; that is, the rectangle contained between NF and FC is equal to double the rectangle contained between CE and LQ. But the rectangle contained between NF and FC is † equal to the square of MF, therefore the square of MF is equal to double the rectangle contained between CE and LQ; but LQ is † equal to GK, and MF is equal to † 36 E. 1. 3.
AH, therefore the square of AH is equal † 34 E. 1. 1.
to double the rectangle contained between

GK and CE; that is, the square of the perpendicular AH is equal to the rectangle contained between the absciss GK, and twice CE, or the *latus rectum* to the axis.

Therefore, *If from any point in the periphery of a parabola a right line be drawn perpendicular to any diameter, and from the same point an ordinate be also drawn to the same diameter, then is the square of the perpendicular equal to a rectangle contained between the absciss cut off by that ordinate, and the latus rectum of the axis; which was to be demonstrated.*

Cor. I. From hence it follows, that the square of the perpendicular lines to the same diameter, drawn from different points of a parabola, are to one another as the respective abscisses, which are made by ordinates drawn to the same diameter from these different points in the parabola; because the squares of these perpendiculars are equal to a rectangle contained between the *latus rectum* of the axis and their respective abscisses; and these rectangles having all the same height, *viz.* the *latus rectum*

Sum of the axis, * are to one another *1 E. 1. 4
as their bases, *viz.* the abscisses; there-
fore *the squares of the perpendiculars*
are as the abscisses.

Cor. 2. If from the vertexes of any two
diameters, ordinates to these diameters
be drawn, the abscisses between these
ordinates and the vertex are equal to
one another; because perpendiculars
drawn from these vertexes are equal to
one another.

PROPOSITION XIV.

THEOREM XIII.

*The squares of all ordinates to the same
diameter are to one another as their
respective abscisses.*

LET RS and AK be ordinates to the
same diameter GH; I say, the square
of the ordinate RS will be to the square of
the ordinate AK, as the absciss GS is to
the absciss GK.

For let the perpendiculars RX and AH
be drawn to the diameter GH; then be-
cause the angles RXS and AHK are right,
and the angles RSX and AKH are * equal, *29 E. 1. 2
the

† 4 E. 1. 6. the triangles RSX and AKH are † similar;
 and therefore as RS is to RX, so is AK
 to AH; and consequently the square of
 † 22 E. 1. 6. RS is † to the square of RX as the square
 of AK is to the square of AH. By alter-
 nation, as the square of RS is to the square
 of AK, so is the square of RX to the square
 of AH: But the square of RX is to the square
 * cor. 1. pr. of AH * as the absciss GS is to the absciss
 † 3 of this. GK; therefore the square of the ordinate
 RS is to the square of the ordinate AK, as
 the absciss GS is to the absciss GK,

Therefore, *The squares of all ordi-
 nates to the same diameter are to one
 another as their respective abscisses;*
 which was to be demonstrated.

PROPOSITION XV.

THEOREM XIV.

*If from any point in a parabola an ordi-
 nate be drawn to any diameter, the
 square of the semi-ordinate is equal to
 a rectangle contained between the ab-
 sciss and the latus rectum of the dia-
 meter.*

LET

Let AB be any parabola, AD a diameter, and EF its directrix, and from the point G in the parabola let GD be drawn a semi-ordinate to the diameter AD , and cutting the axis in I , and let the diameter AD produced cut the directrix in E : I say, the square of DG is equal to the rectangle contained between the abscissa AD and four times AE , or the *latus rectum* of the diameter AD ; for through A the vertex of the diameter AD draw AH parallel to the semi-ordinate GD , and cutting the axis in H , and AK perpendicular to the axis, and through B the principal vertex draw BL a semi-ordinate to the diameter AD .

Then because AH is drawn through the vertex of the diameter AD parallel to the semi-ordinate BL , the line AH is a ^{*}tangent; ^{* cor. 6. pr. 12 of this.} and because $AHBL$ is a parallelogram, AH is equal to BL , and AL is [†]equal to HB . ^{† 34 E. l. 12}

Again, because AH is a tangent to the parabola, and cutting the axis in H , and from the point of contact A , AK is drawn perpendicular to the axis, therefore HB is [†]equal to BK ; and because the triangle ^{† 8 of this.} AKH is right angled at K , the square of AH is ^{*}equal to the sum of the two ^{* 47 E. l. 12} squares described upon AK and KH : But
the

*_{13 of this.} the square of AK is * equal to the rect-

AL angle contained between ~~AK~~ and four times BF; that is, four times the rectangle contained between AL or BH, and BF, and

† 4E. 1. 2. the square of KH, is † equal to four times the square of BK; therefore the square of AH is equal to four times the rectangle contained between BH and BF, together with four times the square of BK; that is, the square of AH is equal to four times the rectangle contained between BK and BF, together with four times the square of BK. But because the line FK is any how cut in B, the rectangle contained between FK and

‡ 3E. 1. 2. KB is ‡ equal to the rectangle contained between KB and BF, together with the square of BK; consequently four times the rectangle contained between FK and KB is equal to four times the rectangle contained between KB and BF, together with four times the square of BK; therefore the square of AH is equal to four times the rectangle contained between FK and KB. But AH is equal to BL, and FK to EA, and BK to EH, which is equal to AL; consequently the square of BL is equal to four times the rectangle contained between EA and AL; that is, the square of the semi-ordinate

BL

BL is equal to the rectangle contained between the absciss AL, and four times AE, which is the *latus rectum* of the diameter AD.

Again, because BL and GD are semi-ordinates to the same diameter AD, the square of BL * is to the square of GD, as the ^{*14 of this.} absciss AL is to the absciss AD: But as AL is to AD †, so is the rectangle con- † 1 E. 1. 6. tained between AL, and four times AE, to the rectangle contained between AD and four times AE; therefore as the square of BL is to the square of DG, so is the rectangle contained between AL, and four times AE, to the rectangle contained between AD and four times AE. But, by the demonstration, the square of BL is equal to the rectangle contained between AL and four times AE; therefore the square of the semi-ordinate GD is † equal † 14 E. 1. 5. to the rectangle contained between AD and the *latus rectum* of the diameter DAE.

Therefore, *If from any point in a parabola an ordinate be drawn to any diameter, the square of the semi-ordinate is equal to a rectangle contained between the absciss and the latus rectum*

F of

of that diameter ; which was to be demonstrated (a).

PROPOSITION XVI.

THEOREM XV.

If from any point a right line be drawn cutting any diameter below its vertex, and parallel to an ordinate drawn to the same diameter ; and if the square of that line be equal to a rectangle contained between the absciss and the latus rectum of that diameter, that point will be a point in the periphery of a parabola.

LET AB be any parabola, C its focus, and EF its directrix ; and let DE be any diameter whose vertex is A, and cutting the directrix in E ; and from the point B in the parabola let BL be drawn any ordinate to the diameter AD ; and if from any point, as G, DG be drawn parallel to

(a) From the property of the squares of the semi-ordinates of any diameter being equal to the rectangles contained between their abscisses and the *latus rectum* of the diameter, it was, that *Appollonius* called this curve a *parabola*.

to BL, and cutting the diameter AD in D below its vertex, and if the square of GD be equal to a rectangle contained between DA and four times EA; I say, G is a point in the periphery of the parabola.

For because the point D * is a point * 4 of this within the parabola, a line drawn through that point produced † will meet the peri- † 10 of this phery of the parabola, if it does not meet it in the point G; if possible, let it meet the periphery of the parabola either nearer or further from the diameter than the point G, as in M.

Then because M is a point in the periphery of a parabola, MD will be a semi-ordinate to the diameter ED; therefore the square of MD is ‡ equal to the rectangle ‡ 15 of this contained between the absciss AD and four times EA. But the square of GD is equal to the rectangle contained between AD and four times EA; therefore the square of GD is equal to the square of MD, a part to the whole, which is absurd; therefore DG cannot meet the periphery of the parabola in any other point than G; consequently G is a point in the periphery of the parabola.

Therefore, *If from any point a right*

F 2

line

line be drawn cutting any diameter below its vertex, and parallel to an ordinate, and if the square of that line be equal to a rectangle contained between the absciss and latus rectum of that diameter, that point is a point in the periphery of the parabola; which was to be demonstrated.

PROPOSITION XVII.

THEOREM XVI.

If from two points, one of which is in the periphery of a parabola, two right lines be drawn cutting a diameter below its vertex, and parallel to ordinates of that diameter, and if the square of these lines be to one another as their respective abscisses, then will the other point be in the periphery of the parabola.

LET AB be any parabola, C its focus, EF its directrix, AD any diameter, which produced cuts the directrix in E, and let B be a point taken in the periphery of the parabola, and G another point, and from the two points B and G let the
lines

lines BL and GD be drawn cutting the diameter in the points L and D, which are below the vertex A, and parallel to the ordinates of the diameter AD; and if the square of BL be to the square of GD, as the absciss AL is to the absciss AD, I say, the other point G is in the periphery of the parabola.

For because the square of BL is to the square of GD, as the absciss AL is to the absciss AD, and four times the line EA is * equal to the *latus rectum* of the diame- * *def. 8 of this.*
ter AD; therefore four times the rectangle † contained between EA and AL † is to four † *E. l. 6.*
times the rectangle contained between EA and AD, as the absciss AL is to the absciss AD; wherefore the square of BL is † to the square of GD, as four times the † *E. l. 5.*
rectangle contained between EA and AL is to four times the rectangle contained between EA and AD. But the square of BL is * equal to four times the rectangle con- * *15 of this.*
tained between EA and AL, therefore the square of GD is equal to four times the rectangle contained between EA and AD; and consequently the point G is a † point * *16 of this.*
in the periphery of the parabola.

Therefore, *If from two points, one of which*

which is in the periphery of a parabola, two right lines be drawn cutting a diameter below its vertex, and parallel to ordinates of that diameter, and if the square of these lines be to one another as their respective abscisses, the other point is in the periphery of the parabola; which was to be demonstrated.

PROPOSITION XVIII.

THEOREM XVII.

If from any point in the periphery of a parabola an ordinate be drawn to any diameter, and from the same point a tangent be drawn and produced until it cut the diameter, the segment of the diameter intercepted between the tangent and ordinate will be bisected in its vertex.

LET A be any point in the parabola AB, and BC a diameter, and from the point A let AC be an ordinate to the diameter BC, and AD a tangent drawn from the same point A, which produced cuts the diameter AC in D; I say, the segment DC, which is intercepted between the tangent

gent

gent and ordinate is bisected in the point B, which is the vertex of the diameter; for through the point A let the diameter AE be drawn, and from the vertex B draw BE an ordinate to the diameter AE.

Then because BE is an ordinate to the diameter AE, it will be * parallel to the tangent AD, and consequently ADBE is a parallelogram; therefore AE is † equal †^{34 E. I. 1.} to BD; but because AE and BC are two diameters, and from their vertexes A and B the ordinates AC and BE are drawn to each other, therefore the absciss AE is ‡ equal to the absciss BC, and consequently BC is equal to BD.

Therefore, *If from any point in the periphery of a parabola an ordinate be drawn to any diameter, and a tangent from the same point, and produced until it cut the diameter, the segment of the diameter, intercepted between the tangent and ordinate, is bisected in the vertex of the diameter; which was to be demonstrated.*

Cor. From hence it follows, that if the line AC be an ordinate to the diameter BC, and the line AD cutting the diameter

meter in D, so as to bisect the segment DC intercepted between that line and the ordinate in the vertex B, that line AD is a tangent to the parabola.

For if AD be not a tangent, let AF, if possible, be a tangent; consequently FB will be equal to BC, (by the above proposition) and likewise FB will be equal to DB, a part to the whole, *which is absurd.*

PROPOSITION XIX.

THEOREM XVIII.

If from any point in the periphery of a parabola an ordinate be drawn to any Diameter, and also a tangent which is produced until it cut that diameter, and from the same point a line is drawn perpendicular to the same diameter; then will the square of the tangent be to the square of the semi-ordinate, as the segment of the diameter, intercepted between the directrix and perpendicular, is to a fourth part of the latus rectum of that diameter.

Case 1. **W**HEN the diameter is the axis.

Lct

Let AB be any parabola, BC its axis, cutting the directrix DE in the point E; and from the point A in the periphery let AC be drawn perpendicular to the axis BC, cutting it in the point C, which in this case is a semi-ordinate; and from the same point A let the tangent AF be drawn cutting the axis BC in F: I say, the square of the tangent AF is to the square of the ordinate AC, as the segment EC of the axis, intercepted between the directrix DE and the perpendicular AC, is to EB, which is one fourth part of the *latus rectum* of the axis.

For because from the point A in the parabola the tangent AF is drawn cutting the axis in F, and from the same point A, AC is drawn perpendicular to the axis; therefore the segment of the axis FC is * bise- * 8 of this cted in B; and because the triangle ACF is right angled at C, the square of AF is † equal to the two squares of AC and CF. † 47 E. I. 12 But the square of AC is ‡ equal to four ‡ 13 of this times the rectangle contained between EB and BC, and the square of FC is * equal * 4 E. I. 24 to four times the square of BC; therefore the square of AF is to the square of AC, as four times the rectangle contained be-

G

tween

tween EB and BC, together with four times the square of BC, is to four times the rectangle contained between EB and BC. But four times the rectangle contained between EB and BC, together with four times the square of BC, is to four times the rectangle contained between EB and BC, as
 * E. l. 6. the base EC is to the base EB *; therefore the square of the tangent AF is to the square of the semi-ordinate AC, as the segment of the axis EC, which is intercepted between the directrix DE and the perpendicular AC, is to EB one fourth of the *latus rectum*; W. W. D.

Case 2. When the diameter is not the axis.

Let ACB be any parabola, C its principal vertex, and FG its directrix; let FKB be any diameter, and from any point in the periphery, as A, let AE be drawn a tangent, cutting the diameter FK in E, and from the same point A let AK be drawn perpendicular to the diameter FK, and AD a semi-ordinate to FK: I say, the square of the tangent AE is to the ^{square} ~~rectangle~~ of the semi-ordinate AD, as FK the ^{segment} ~~square~~ of the diameter, intercepted between the directrix and perpendicular, is to FB the fourth part

part of the *latus rectum* of the diameter BK.

For through the principal vertex C * draw ^{* 31 E. I. 1.} CH parallel to the directrix FG, cutting the diameter BK in H.

Then, because CH is parallel to FG, and FH parallel to GC, FH is † equal to GC, ^{† 34 E. I. 1.} that is, one fourth of the *latus rectum* of the axis. Again, because AE is a tangent drawn from the point A cutting the diameter BK in E, and AD a semi-ordinate drawn from the same point A, cutting the diameter in D, therefore ED is † bisected in B; ^{† 18 of this.} and because the angle AKE is a right angle, the square of AE is * equal to the two ^{* 47 E. I. 1.} squares upon AK and KE, and the square of AD is equal to the two squares upon AK and KD. But the square of AD is † equal to four times the rectangle contained between FB and BD, ^{† 15 of this.} and the square of AK is † equal to four times the rectangle contained between FH and BD; ^{† 13 of this.} therefore the square of DK will be equal to four times the rectangle contained between HB and BD. Again, because KB is any how cut in D, and BE equal to BD is added to it, therefore the square of EK is * equal ^{* 8 E. I. 2.} to four times the rectangle contained be-

tween KB and BD, together with the square of DK; that is, the square of EK is equal to four times the rectangle contained between KH and BD; consequently the two squares described upon AK and KE, that is, the square of AE, are equal to four times the rectangle contained between FK and DB; therefore the square of AE is to the square of AD, as four times the rectangle contained between FK and BD is to four times the rectangle contained between FB and BD. But four times the rectangle contained between FK and BD is to four times the rectangle contained between FB and BD, * as FK is to FB; therefore the square of the tangent AE is to the square of the semi-ordinate AD, as FK the segment of the diameter, intercepted between the directrix FG and the perpendicular AK, is to FB the fourth part of the *latus rectum* of the diameter BK.

Therefore, *If from any point in the periphery of a parabola an ordinate be drawn to any diameter, as also a tangent cutting the diameter, and a line perpendicular to the same diameter, the square of the tangent is to the square of the semi-ordinate, as the segment of the dia-*

diameter, intercepted between the directrix and the perpendicular, is to the fourth part of the latus rectum of that diameter; which was to be demonstrated.

Cor. 1. From hence it follows, that the square of a perpendicular to any diameter, drawn from a point in the periphery of a parabola, is to the square of a tangent drawn from the same point to the diameter, as the fourth part of the *latus rectum* of the axis is to the segment of the diameter intercepted between the directrix and perpendicular.

For because the square of the tangent AE (by the above demonstration) was proven equal to four times the rectangle contained between FK and BD, and the square of AK is equal to four times the rectangle contained between GC and BD; therefore the square of the perpendicular AK is to the square of the tangent AE, as four times the rectangle contained between GC and BD is to four times the rectangle contained between FK and BD, that is, as GC is to FK; wherefore the square of the perpendicular AK is to the square of the tangent AE, as GC, the fourth part of the *la-*
tus

latus rectum of the axis is to FK the segment of the diameter between the directrix and the perpendicular.

Cor. 2. If from any point in the periphery of a parabola an ordinate be drawn to any diameter, and a line perpendicular to the same diameter, the square of the segment of the diameter, intercepted between the ordinate and perpendicular, is equal to the rectangle contained between the absciss and a segment of the same diameter, intercepted between the vertex of the diameter, and a line drawn cutting it from the principal vertex parallel to the directrix.

Cor. 3. If from any point in the periphery of a parabola a tangent be drawn cutting any diameter, the square of the tangent intercepted between the point of contact and the point where it cuts the diameter, is equal to a rectangle contained between that segment of the diameter intercepted between its vertex and the tangent, and the *latus rectum* of a diameter drawn through the point of contact.

For because (by the proposition) the square of AE is equal to four times the rectangle
angle

angle contained between BD and FK, and BD is equal to BE, and FK is equal to LA, therefore the square of the tangent AE, intercepted between the point of contact A and the diameter ED, is equal to the rectangle contained between BE, the segment intercepted between the vertex and the tangent, and four times LA, which is the *latus rectum* of the diameter drawn through the point of contact A.

PROPOSITION XX.

PROBLEM II.

To find a diameter, the axis, the latus rectum, and the focus of any parabola which is given in position.

LET ABCD be any parabola; it is required to find its diameters, its axis, its *latus rectum* and its focus.

First, for a diameter. Take any two points in the periphery of the parabola, as AB, and through these points A and B draw AD and BC * parallel to one ano- * 31 E.L.I.
ther; and cutting the periphery of the parabola again in the points B and C, † bi- † 10 E.L.I.
sect

sect the two lines BC and AD in the points E and F, and join the line FE: I say, the line FE produced is a diameter.

For because the points E and F are within the parabola, the line EF, if produced, ^{*11 of this.} will * cut the parabola, which let be in the point G.

Then because the right line GEF is drawn from any point of the periphery of the parabola, bisecting BC and AD, which are parallel to one another, and terminated both ways by the periphery, the line GF is a † diameter to the parabola; therefore there is found a diameter to the given parabola ABCD; *which was to be done.*

† cor. 5. pr.
22 of this.

Secondly, A diameter, as GF, being found, it is required to find the axis.

Take any point in the diameter FG below its vertex, as H, and from H draw HI ^{‡11 E. l. 1.} ‡ perpendicular to FH, which let be produced both ways until it cut the periphery of the parabola in I and K; ^{*10 E. l. 1.} * bisect IK in the point L; and through L draw ^{†31 E. l. 1.} † LM † parallel to the diameter GF, meeting the parabola in M, and from the point M let MN be drawn parallel to IL.

Then because ML is parallel to GF, ^{‡ def. 4 of this.} ‡ ML is a † diameter, and the line IK is an

* or-

* ordinate to it; and because through the vertex M of the diameter LM the line MN is drawn parallel to its ordinate IL, therefore the line MN is a † tangent to the parabola; and because the angle LMN is a right angle to the diameter, ML is the † axis: Hence there is found the axis of a given parabola; *which was to be done.*

* def. 3 of this.
† cor. 6. pr. 12 of this.
† cor. 2. pr. 7 of this.

Thirdly, The axis being found, to find the *latus rectum* and focus.

Find a * third proportional to the two lines ML and LI, which let be O, the line O will be the † *latus rectum* of the axis, and the fourth part of O will give the † focal distance from the vertex of the axis; therefore there will be found the *latus rectum* of the axis and the focus; *which was to be done.*

* 11 E. 1.6;
† 15 of this.
† def. 8. cor. pr. 1 of this.

After the same manner, if the axis be produced from its vertex, and a part cut off equal to the focal distance, such as MP, and through the point P a right line, as PQ, be drawn at right angles to the axis, the line PQ will be the directrix of the parabola.

H

P R O-

PROPOSITION XXI.

PROBLEM III.

With a given right line, and a point out of it, to describe a parabola, which will have the given line for its directrix, and the given point for its focus.

LET AB be the given right line, and C the given point, it is required to describe a parabola, which will have AB for its directrix and C its focus.

First, It is plain that the parabola may
** def. 2. of this.* be * described by the assistance of a square, and a thread fixed to it of the same length with one of the sides of the square; or any point in the periphery of the parabola may be found by the following Construction.

‡ 12 E. 1. 1. Through the point C draw CB ‡ perpendicular to AB, and take any point in it, as D, and draw DE perpendicular to BD, and from the point C in the line CD produced cut off a part, as CG, equal to BD, and about the center C, with the distance CG, describe a circle cutting the line ED in the points E and F: I say, the points

points E and F are in the periphery of the parabola, which has the point C for its focus, and AB its directrix.

For from the point E draw EA * paral- * 31 E. l. 1.
lel to BD, and join EC.

Then because C is the focus, and CB is drawn at right angles to the directrix AB, CB † is the axis; and because AE is † ^{def. 4 of this.} parallel to BC, and ED is parallel to AB, AE is † equal to BD. But BD is (by the † 14 E. l. 1. construction) equal to CG, that is to CE, therefore AE is equal to CE, and consequently E is * a point in the periphery of * 2 of this; the parabola.

And after the same manner may it be proven, that F is a point in the periphery of the parabola; and by repeating the above operation any number of points may be found in the periphery of a parabola, and by joining these points the parabola will be described, which has the point C for its focus, and the line AB for its directrix: Therefore there is described a parabola, which has the given right line for its directrix, and the given point out of it for its focus; *which was to be done.*

Cor. From hence it follows, that a parabola may be described, the line of direction and vertex of the axis being given; or the vertex of the axis and focus given; or if the axis its vertex, and *latus rectum* be given; and lastly, if the axis, the focus, and *latus rectum* of the axis be given.

Case 1. If the directrix AB and vertex of the axis H be given: From the point ^{*12 E. l. 1.} H draw HB * perpendicular to AB, and make HC equal to HB, then will the point C be the † focus; then by the ^{† cor. pr. 1 of this.} above proposition may the parabola be described, whose focus is C and directrix AB.

Case 2. If the focus be C, and vertex of the axis H: Join HC, and make HB equal to HC, and from the point B draw BA at right angles to BC, then will BA be the directrix, and the parabola whose focus is C and directrix AB, be described as before.

Case 3. If HD be the axis, the point H its vertex, and the line IK its *latus rectum*; Let IL be taken ‡ equal to the ^{‡ 9 E. l. 6.} fourth part of IK, and make HB and HC each equal to IL, and from B draw
BA

BA perpendicular to BC, then will C be the * focus, and AB the directrix; * *def. 9 of this.* and the parabola, whose focus is C, and directrix AB, be described as before.

Case 4. If HD be the axis, C the focus, and IK the *latus rectum* of the axis: Take IL the † fourth part of IK, make † 9 E. 1. 6. HC and HB each equal to IL, and from the point B draw BA † perpendicular to † 12 E. 1. 1. BC, then will H be the vertex of the parabola, and AB the directrix, and the parabola be described as before.

PROPOSITION XXII.

PROBLEM IV.

A right line and a point in it being given, as also a point out of it, which is below the point in the given line, to describe a parabola which will have the given line for its axis, the given point in the line for its principal vertex, and which will pass through the other point.

LET the given right line be AB, the given point in it A, and C the given point out of the line AB, and below the point

point A; it is required to describe a parabola having the line AB for its axis, the point A its principal vertex, and which will pass through the point C.

For from the point C let CD be drawn
 * 12 E. I. 1. at * right angles to AB, and find EF a
 † 11 E. I. 6. † third proportional to AD and DC, and
 ‡ 9 E. I. 6. take EG ‡ equal to a fourth part of EF,
 and make AH and AI each equal to EG,
 ‡ 31 E. I. 1. then from the point H draw HK * parallel
 to CD, and describe a parabola, such
 as LAM, whose focus is I and directrix
 HK; I say, that parabola will pass through
 the given point C.

For because AD is to DC as DC is to
 EF, the rectangle contained between AD
 † 17 E. I. 6. and EF is † equal to the square of CD,
 and EF is the *latus rectum* of the diameter
 AB, consequently the point C is ‡ in the
 periphery of the parabola LAM; therefore
 there is described a parabola which has the
 given line AB for its axis, the given point
 in it A for its principal vertex, and which
 passes through the other given point C,
 which is out of the given line, and below
 the given point; *which was to be done.*

‡ cor. pr. 16
 of this.

PRO-

PROPOSITION XXIII.

PROBLEM V.

Two right lines being given, mutually cutting one another, and a third line of any magnitude being also given, to describe a parabola which shall have one of the given lines cutting one another for a diameter, the other a tangent, and the third line given of any magnitude for the latus rectum or parameter of that diameter.

LET AB and AC be the two given lines, cutting one another at the point A, and DE the other given finite right line; it is required to describe a parabola which will have AB for a diameter, AC a tangent touching it in the point A, and the line DE for the *latus rectum* or parameter of the diameter AB.

Take DF, * equal to the fourth part of ⁹E. I. 6; DE, and produce AB, and make AG equal to DF, and through G draw GH at † right angles to GA, and make the angle † ¹¹E. I. 1. CAI † equal to the angle CAG, and the † ²³E. I. 1. right line AI equal to AG; then describe the * parabola AKL, whose focus is I and ²² of this. di-

directrix GH, then will AB be a diameter
 * *def. 4 of this.* whose *latus rectum* is * four times AG,
 that is DE; and because the angle GAC
 is equal to the angle CAI, the right line
 † *s of this.* AC will be a † tangent to the parabola;
 therefore there is described a parabola
 AKL, which has one of the lines AB cut-
 ting one another for a diameter, the other
 AC a tangent, and the third given line
 DE of any magnitude for the *latus re-*
ctum of the diameter AB; *which was*
to be done.

PROPOSITION XXIV.

PROBLEM VI.

With a given right line and a given point in it, as also any other line given in magnitude and position cutting the first given line in a point below the first given point, to describe a parabola which will have the first given line for a diameter, the given point in it the vertex of that diameter, and which will pass through the extremity of the other given line in such a manner as that line may be a semi-ordinate to the diameter.

LET

LET AB be the given right line, and A a given point in it, and CD any other line given, cutting the line AB in D below the point A; it is required to describe a parabola which will have AB for a diameter, A its vertex, and to pass through the point C, so as to make CD a semi-ordinate to the diameter AB.

Through the point A draw AE * paral- * 31 E. I. 12
 lel to CD, and find a † third proportional † 9 E. I. 6,
 to AD and DC, which let be FG, and
 ‡ describe the parabola AHI, which has † 23 of this
 AB for a diameter, its *latus rectum* FG,
 and AE a tangent to it touching it in the
 point A: I say, the parabola will pass
 through the point C, and CD will be a
 semi-ordinate to the diameter AB.

For because the line DC is parallel to
 the tangent AE, DC is a * semi-ordinate * 12 of this;
 to the diameter AB; and because AD is
 to DC as DC is to FG, the rectangle con-
 tained between AD and FG is † equal to † 17 E. I. 64
 the square of DC; and because from the
 point C the line CD is drawn to the dia-
 meter AB, parallel to an ordinate, and
 cutting it below its vertex A, so as to
 make the rectangle contained between the
 absciss AD and *latus rectum* FG equal to

the square of the line CD, the point Q
 *16 of this. * is in the periphery of the parabola;
 therefore there is a parabola AHI describ-
 ed, which has the given line AB a diam-
 eter, the given point A in it, its vertex, and
 the other given line DC cutting AB below
 the point A, a semi-ordinate to AB, and pas-
 sing through C the extremity of the other
 line DC; *which was to be done.*

The following definitions are the first nine
 in the first Book of *Apollonius Per-
 geus's Conics.*

I, 10. *If from any point a right line
 be drawn touching the circumference of
 a circle, which is not in the same plane
 with it, and produced both ways, the
 point being fixed, and the right line
 moving round the circumference of
 the circle until it come to the place
 it began to move at; the surfaces de-
 scribed by the motion of the right line,
 and having the fixed point for both
 their vertexes, either of which sur-
 faces may be infinitely augmented, if
 the right line with which they are ge-
 nerated be infinitely produced; is called
 a conical surface.*

2, 11. The fixed point round which the right line moves is called the vertex.

3, 12. A right line which is drawn through that point and the center of the circle is called the axis.

4, 13. The figure which is contained between the circle and the conical surface, intercepted between the vertex and the circumference of the circle, is called a cone.

5, 14. The vertex of the cone is that point which is the vertex of the conical surface.

6, 15. The axis of the cone is a right line which is drawn from its vertex to the center of the circle.

7, 16. The circle itself is called the base of the cone.

8, 17. A cone which has its axis perpendicular to the base is called a rectangular cone.

9, 18. Those cones are called scalens, which have not their axis perpendicular to their bases.

PROPOSITION XXV.

THEOREM XIX.

Right lines drawn from the vertex of the surface of a cone to points that are in the surface, coincide with the surface of the cone (a).

LET BAC be the surface of any cone, its vertex A, and the right line ADB, joined to the point B in the surface; I say, the right line ADB will wholly coincide with the surface; for, if possible, let it not coincide with the surface, and let CE be the right line which described the surface, and let BFC be the circle round which EC is moved; therefore if A remain fixed, and EC move round the circle BFC, then will the line EC pass through the point B, and consequently there will be two different right lines ADB and AGB, having the same

* ax. 10 E. extremities, which is * absurd.
h. t.

Therefore *right lines drawn from the vertex of a cone to points in the surface, will wholly coincide with the surface; which was to be demonstrated.*

Cor.

(a) This is the first proposition of the first book of Apollonius Pergaeus.

Cor. From hence it is manifest, that if a right line be drawn from the vertex of the surface of a cone, to a point either within or without the cone, in the first case it will ly wholly within the cone, and in the second case wholly without it.

PROPOSITION XXVI.

THEOREM XX.

If a cone be cut by a plane passing through its vertex, the section is a right lined triangle (a).

LET ABC be any cone, its vertex A, and its base the circle BCF, which is cut by any plane passing through the point A, and making a section that is bounded in the surface with certain lines, as AH and AK, and in the base with the line HK, which last line is a * right line: I say, *E. I. 11; AHK is a right lined triangle.

For because a right line drawn in the cutting plane, from the point A to the point H, † coincides with the surface of the cone, therefore the cutting section, and
fur-

(a) This is the third proposition of the first book of Apollonius.

surface of the cone will coincide in the same common section, and consequently the lines AH and AK, which are their common sections, are ^{*}right lines. But HK was before proven to be a right line; wherefore the triangle AHK is a right lined triangle.

Therefore, *If a cone be cut by a plane passing through its vertex, the section is a right lined triangle; which was to be demonstrated.*

PROPOSITION XXVII.

THEOREM XXI.

If either of the conical surfaces about the same vertex be cut by a plane parallel to the circle about which the right line moved that described the surface, the section bounded by the surface is a circle which has its center in the axis; and that figure lying between the circle, and that part of the conical surface intercepted between its vertex and the cutting plane, is a cone (a).

LET

(a) This is the fourth proposition of the first book of Apollonius.

[ET there be any conical surface whose vertex is A, and BDC the circle in which the line is carried that describes the surface, and let it be cut by any plane parallel to the circle BDC, and let the section make in the surface a figure bounded by the line EFL; I say, the figure EHFL is a circle which has its center in the axis.

For find * I the center of the circle BDC, * 1 E. 1. 4. and join AI, then will AI † be the axis, † def. 12 of this. and will meet the cutting plane, which let be in the point K, and through the axis AI let any plane be drawn ABC, the section made by that cutting plane, viz. ABC, is a † right lined triangle; and be- † 26 of this. cause the points E, K, F, are in the cutting plane ELF, and also in the plane ABC, the line EKF is a * right line. Again, let * 3 E. 1. 11; any point, as H, be taken in the line DLE, join AH, which produced will meet the circumference of the circle BDC, which let be in D, and join KH and ID.

Then because the two parallel planes BDC and ELF are cut by a third ABC, their common sections † are parallel, there- † 16 E. 1. 11; fore BC is parallel to EF; and for the same reason ID is parallel to KH; consequently the two triangles AIB, AKE, are † similar; † 4 E. 1. 6. there-

therefore as AK is to AI, so is EK to BI,
 and for the same reason as AK is to AI,
 so is KH to ID; and also as AK is to AI,
 * 11 E. 5. so is KF to CI; therefore * as IB is to KE,
 so is ID to KH, and so is IC to KF. But
 BI, ID and IC, are all equal to one another,
 † 14 E. 1. † therefore EK, KH and KF, are † likewise
 equal to one another; and after the same
 manner may it be proven, that all the
 right lines drawn from the point K to the
 line EHFL are equal to one another; there-
 fore the figure EHFL is a circle, and has
 its center K in the axis; and since the fi-
 gure EHFL is a circle, it is evident that
 the figure intercepted between that circle
 and the vertex A is a cone.

Therefore, *If either of the conical sur-
 faces about the same vertex be cut by a
 plane parallel to the circle about which
 the right line moved that described the
 surface, the section bounded by the sur-
 face is a circle that has its center in
 the axis; and that figure lying between
 the circle and that part of the conical
 surface, intercepted between its vertex
 and the cutting plane, is a cone; which
 was to be demonstrated.*

PROPOSITION XXVIII.

THEOREM XXII.

If a cone be cut by a plane passing through the axis, as also by another plane which cuts the base of the cone in the direction of a right line, perpendicular to the base of that triangle passing through the axis, and if the common section of the cutting plane; and of the triangle passing through the axis, be parallel to one side of the same triangle; the figure formed by the common section of the cutting plane and the conical surface, will be a parabola having the common section of the cutting plane, and the triangle passing through the axis for a diameter.

LET A be the vertex of a cone, the
the circle BDC its base cut by a plane
through its axis, and let the triangle ABC
be formed by that section, and cut by an-
other plane which cuts the base of the
cone, according to the direction of the
right line EG perpendicular to BC, and
K let

let EFG be the figure which the section makes with the conical surface, and let FM, the common section of the cutting plane and triangle through the axis, be parallel to AC one side of the same triangle: I say, the figure formed by the line EFG is a parabola, which has FM the common section of the cutting plane and triangle for one of its diameters.

For take any point in the section GFE, ^{*31 E. l. 1.} such as H, and through H draw HI ^{*} parallel to GE, and produce it until it cut the line MF in I, and through I draw KIL parallel to BC.

Then because LI is parallel to BC, and HI to EG, the plane which passes through ^{†15 E. l. 11.} LI and IH is [†] parallel to the plane which passes BC and EG; that is, the plane passing through LI and IH is parallel to the base of the cone; therefore the plane passing through LI and IH [‡] is a circle whose diameter is LK; and because EG is perpendicular to BC, therefore HI is ^{*} perpendicular to LK; therefore the rectangle contained between IL and IK is [†] equal to the square of HI; and for the same reason the rectangle contained between BM and MC is equal to the square of ME; therefore

fore the square of IH is to the square of ME, as the rectangle contained between IL and IK is to the rectangle contained between BM and MC; and because IK is * equal to MC, therefore the rectangle contained between IL and IK is to the rectangle contained between BM and MC, † as the base † 1 E. 1. 6. LI is to the base BM; and because the triangles LIF and BMF are similar ‡, as LI is † 4 E. 1. 6. to BM, so is FI to FM; therefore the square of HI * is to the square of EM, as FI is † 11 E. 1. 5. to FM; and if a parabola be † described, † 24 of this, which has FM for its diameter, and F the vertex, and EM an ordinate to that diameter, and because by the construction the point E is in the periphery of the parabola, the point H † will also be a point in † 14 of this; the periphery of the parabola. And after the same manner may it be proven, that all the points in the section are points in the periphery of a parabola, and consequently the figure EFG is a parabola, and FM the common section of the cutting plane, and triangle passing through the axis, is a diameter to it.

Therefore, *If a cone be cut by a plane passing through its axis, and also by another plane which cuts the base in the*

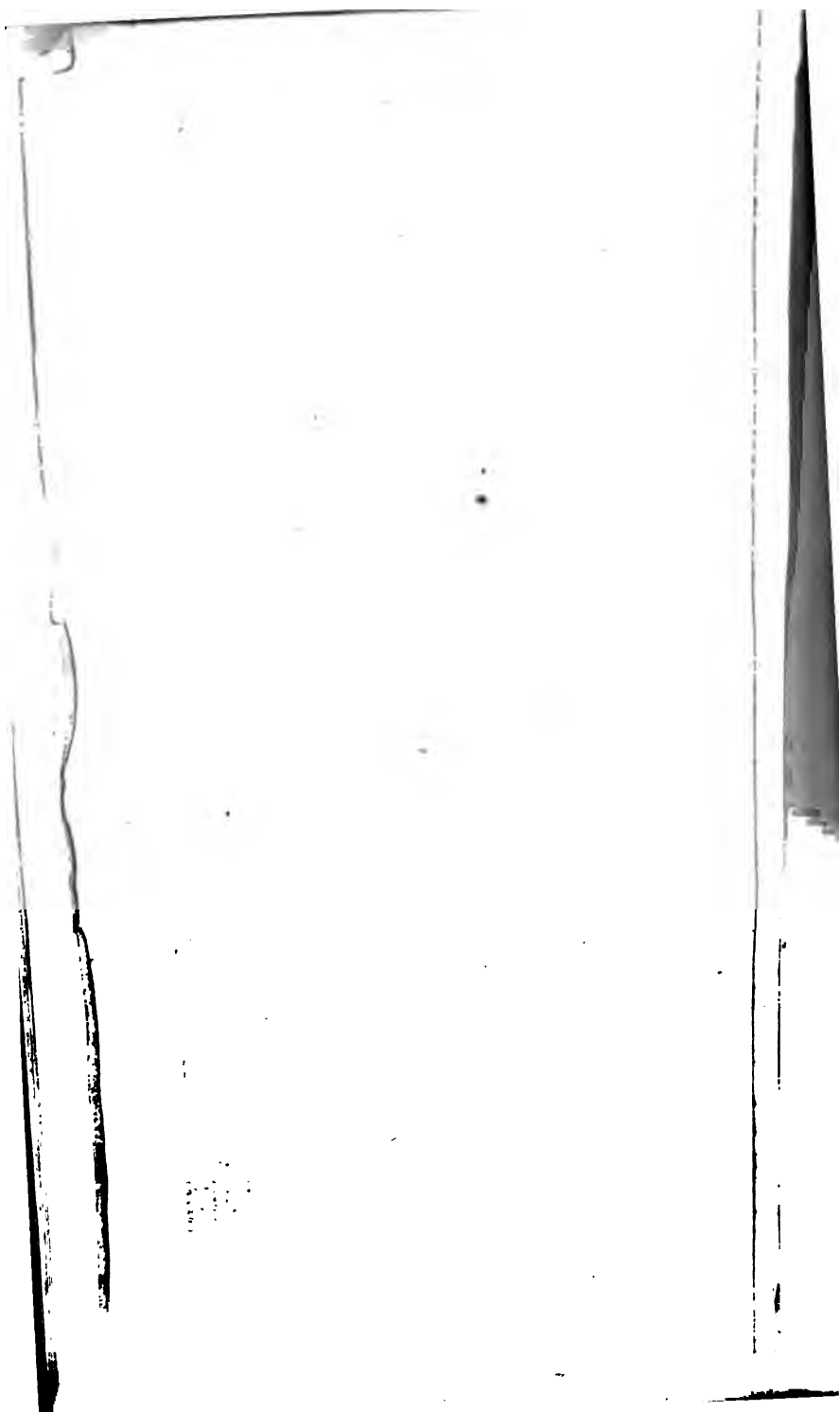
direction of a right line, perpendicular to the base of the triangle passing through the axis, and if the common section of the cutting plane, and triangle passing through the axis, be parallel to one side of the same triangle, the figure formed by the common section of the cutting plane and the conical surface, is a parabola which has the common section of the cutting plane, and the triangle passing through the axis, for one of its diameters; which was to be demonstrated.

The end of the first book.



E L E





ELEMENTS OF Conic Sections,

BOOK II. OF the ELLIPSE.

DEFINITIONS.

I *F any two points, as A and B, be taken in any plane, and in them are fixed the extremities of a reed, whose length is greater than the distance between the points and the reed extended by means of a small pin and if the pin be moved round from any point until it return to the place from whence it began to move, the thread being extended during the whole time of the revolution, the figure which the small pin by this revolution describes is called an ellipse.*

II. *The two points A and B are called the foci.*

III. *The*

III. The point D, where the line is intercepted between the two foci is called the center of the ellipse.

IV. The distance between the center and one of the foci is called the eccentricity of the ellipse.

V. Any right line drawn through the center, and terminated both ways by the ellipse, is called a diameter; and the two points where it meets the ellipse are called the vertexes of that diameter.

VI. The diameter which passes through the foci is called the greater, or transverse axis.

VII. The diameter which cuts the greater axis at right angles is called the lesser axis.

VIII. Two diameters are said to be conjugate, when either of them bisects all the lines that are terminated both ways by the ellipse, and parallel to the other.

IX. Any right line terminated both ways by the ellipse, but does not pass through the center, and is bisected by a diameter, is called an ordinate to that diameter.

Also any diameter which is parallel

to any ordinate of a diameter, is also called an ordinate to that diameter.

X. *The segment of a diameter intercepted between any ordinate and its vertex is called the absciss of that ordinate.*

XI. *A third proportional to two conjugate diameters is called the parameter, or latus rectum to that diameter, which is first in proportion.*

XII. *Any right line which touches the ellipse in one point only, and being produced falls without it, is called a tangent to the ellipse.*

PROPOSITION I.

THEOREM I.

If from any point in the periphery of an ellipse two right lines be drawn to the foci, these two lines taken together are equal in length to the greater axis.

LET C be any point taken in the periphery of an ellipse, and A and B the two foci, and EF the greater axis, and let the two lines CA and CB be drawn from the point to the foci; I say, these

these two lines CA and CB taken together are equal to the greater axis EF.

For because C is a point taken in the periphery of an ellipse, the two lines AC and CB taken together are * equal to the length of the threed with which the ellipse was described; and because the points E and F are also in the periphery of the ellipse, the two lines EA and EB taken together are equal to the two lines AF and BF taken together; because they are both equal to the length of the generating threed, take AB, which is common, away from both, and there will remain twice EA equal to twice BF, and consequently EA is equal to BF; add EB, which is common, then will the two lines EA and EB taken together be equal to EF the greater axis. But EA and EB taken together are equal in length to the generating threed with which the ellipse was described, and AC and CB taken together were also proven equal in length to the same generating threed; therefore AC and CB taken together are equal in length to EF the greater axis of the ellipse.

Therefore, *If from any point in the periphery of an ellipse two right lines*
be

be drawn to the foci, these two lines taken together are equal to the greater axis; which was to be demonstrated.

Cor. 1. From the above demonstration it follows, that the greater axis of an ellipse is bisected in the center; for because * AD is equal to DB, and EA ^{* def. 3 of this.} was proven equal to BF, therefore ED is equal to DF.

Cor. 2. If two right lines be drawn from any point, without an ellipse, to the foci, these two right lines taken together are greater than the greatest axis; but if two right lines be drawn to the foci of an ellipse from any point within it, these two lines taken together are less than the greater axis, both of which plainly follow from *prop. 21. E. l. 1.*

Cor. 3. From hence it is manifest, that if the sum of two right lines taken together, drawn from any point to the foci of an ellipse, be either greater, equal to, or less than the greater axis, in the first case the point will be without the ellipse, in the second it will be in the periphery, and in the last case it will be within it.

Cor. 4. The distance of the vertexes of
L
the

the smaller axis from the foci is equal to half the greater axis; for let G and H be the vertexes of the smaller axis, and join AH and HB, then because AD is * equal to DB, and DH is common, and the angles ADH and BDH are both † right angles, therefore the base AH is ‡ equal to the base BH, and (by the above proposition) both taken together are equal to the greater axis; therefore each of them is equal to half the greater axis.

* cor. 1. pr.
1 of this.

† def. 7 of
this.

‡ 4 E. 1. 1.

Cor. 5. The lesser axis of any ellipse, as GH, is bisected in the center; for from the focus A draw the right lines AH and AG to its vertexes, which are equal to one another, (by the preceding cor.) therefore the angles AGD and AHD are * equal to one another, and the angles ADG and ADH are right angles, consequently GD is † equal to DH; therefore the lesser axis GH is bisected in the center D.

* 5 E. 1. 1.

† 26 E. 1. 1.

PROPOSITION II.

THEOREM II.

The square of half the smaller axis of any ellipse is equal to a rectangle contained between the segments of the greater axis, intercepted between its vertexes and one of the foci.

LET EGF be any ellipse whose foci are A and B, and D its center, EF the greater axis, and GH the smaller; I say, the square of GD, half the smaller axis, is equal to a rectangle contained between FA and AE, the segments of the greater axis, intercepted between its vertexes E and F, and one of the foci A. For join AG;

Then because the angle GDA is a right angle, the square of GA is * equal to the * 47 E. I. 1. sum of the two squares of GD and DA:

But AG is † equal to ED, therefore the † cor. 4. pr. 1 of this. two squares described upon GD and DA

are equal to the square of DE. Again, because the line EF is equally cut in D, and unequally cut in A, the square of ED is ‡ equal to the rectangle contained between ‡ 5 E. I. 2.

FA and AE, together with the square of AD; therefore the two squares described

L 2

upon

upon AD and DG are equal to to the rectangle contained between FA and AE, together with the square of AD; take from both the square of AD, (which is common) there will remain the square of GD, half the smaller axis, equal to the rectangle contained between FA and AE, that is, the segments of the greater axis intercepted between its vertexes and one of its foci.

Therefore, *The square of half the smaller axis of any ellipse is equal to a rectangle contained between the segments of the greater axis intercepted between its vertexes and one of the foci; which was to be demonstrated.*

PROPOSITION III.

THEOREM III.

Every diameter of an ellipse is bisected in the center,

LET KL be a diameter of any ellipse, whose center is D, I say, KD is equal to DL; for if DK is not equal to DL, let DK, if possible, be greater than DL, which produce, and make DI equal to DK, and from the points K, L and I, draw the lines KA,

KA, KB, LA, LB, IA and IB to the foci.

Then because AD is * equal to DB, ^{* cor. 1. pr. 1 of this.} and KD is equal to DI, and the angle KDB † equal to the angle ADI, therefore † 15 E. 1. 1. the base KB is ‡ equal to the base AI; and ‡ 4 E. 1. 1. after the same manner may AK be proven equal to IB; consequently the two lines AK and KB are equal to the two lines AI and IB. But the two lines AK and KB are * equal to the two lines AL and LB, * 1 of this; therefore the two lines AI and IB are equal to the two lines AL and LB; that is, two lines drawn from the extremity of one side of a triangle, meeting within it, equal to the two other sides of the triangle, † which is absurd; therefore DL is † 21 E. 1. 1. not smaller than DK, and after the same manner may it be proven not to be greater than DK; consequently the diameter KL is bisected in the center D.

Therefore, *Every diameter of an ellipse is bisected in the center; which was to be demonstrated.*

[PRO-

PROPOSITION IV.

THEOREM IV.

*Haec propositio quarta nihil pre-
est nisi applicatio
propositionis 14^{ae}
Keillii Trigonometricae
namque sumptis illarum
quantitatum dimidiis
quae ibi proportionales
esse ostenduntur.*

If from any point in the periphery of an ellipse, which is not the vertex of either axis, a right line be drawn to the nearest focus, as also a line perpendicular to the greater axis; and if from the vertex of the axis nearest to the given point, a part be cut off towards the center, equal to the line drawn from the given point to the nearest focus, half the greater axis has the same proportion to the excentricity, as the segment of the axis, intercepted between the center and perpendicular, has to the excess by which half the greater axis exceeds the line drawn from the given point to the nearest focus. Also the square of the perpendicular will be less than the rectangle contained between the segments of the greater axis, intercepted between its vertexes and perpendicular by a rectangle contained between the segments of the same axis, intercepted between the extremity of the line cut off from the

the vertex of the axis towards the center, equal to the line drawn from the given point to the nearest focus, and the foci.

LET ABC be any ellipse, AC its greater axis, D and E the foci, and F the center, and B any point taken in the periphery; and let BE be drawn to the nearest focus, and BG perpendicular to the greater axis AC, and from the vertex C, nearest to the given point B, CH is cut off towards the center equal to BE: I say, as FC, half the greater axis, is to FE the excentricity, so is FG, the distance between the center and perpendicular, to FH the excess by which the semi-axis FC exceeds BE the line drawn from the given point to the nearest focus.

For join BD to the other focus D, and about the center B, with the distance BE, describe the circle IKL, cutting the greater axis again in the point K, and the line DB in the points I and L.

Then because DE is * double to FE, * *def. 3 of this.*
and EK is † double to EG, therefore † *3 E. l. 3.*
DK is double to FG; and because DL is equal to DB, and BE taken together,
and

and DB and BE taken together are * equal to AC, therefore DL is equal to AC, consequently DL is double of FC, and IL is double of IB or CH; therefore the remaining part DI is double the remaining part FH; and because DL and DE are drawn cutting the circle from a point without it, the rectangle contained between DL and DI is † equal to the rectangle contained between DK and ED; therefore as DL is to ‡ DE, so is DK to DI, and, by taking their halves, it will be as FC, half the greater axis, is to FE the excentricity, so is FG, the distance of the center from the perpendicular, to FH the excess by which half the greater axis exceeds CE; that is, BE the line drawn from the given point to the nearest focus. W. W. D.

* 1 of this.
† cor. 1. 37
E. l. 3.

‡ 16 E. l. 6.

I say likewise, that the square of the perpendicular BG will be less than the rectangle contained between AG and GC, *viz.* the segments of the greater axis, intercepted between its vertexes and the perpendicular, by the rectangle contained between DH and HE, *viz.* the segments of the greater axis, intercepted between H the extremity of the line HC, cut off towards the

the

the center from the nearest vertex C, equal to BE the line drawn from the given point to the nearest focus, and the foci D and E.

For because the right line FC is any how cut in H, * the squares described upon FC and FH are equal to twice the rectangle contained between CF and FH, together with the square of HC; but the square of HC is equal to the square of BE, therefore the two squares described upon FC and FH are equal to twice the rectangle contained between FC and FH, together with the square of BE. But the square of BE is † equal to the two squares † 47 E. I. described upon BG and GE, consequently the two squares described upon FC and FH are equal to twice the rectangle contained between FC and FH, together with the two squares described upon BG and GE. Again, because (by the above demonstration) FC is to FE as FG is to FH, therefore the rectangle contained between FC and FH ‡ is equal to the rectangle contained between FG and FE; consequently twice the rectangle contained between FG and FE, together with the two squares described upon BG and GE, are equal to the

M

two

two squares described upon FC and FH.
^{FE} But because the line FG is any how cut in
^{* 7 E. l. 2.} E, * the two squares described upon FG
^{† G} and FE are equal to twice the rectangle
 contained between GF and FE, together
 with the square of EG; consequently the
 two squares described upon FC and FH are
 equal to the three squares described upon
 FG, FE and BG; take from both the two
 squares described upon FG and FH, and
 there will remain the rectangle contained
^{† 5 E. l. 2.} between AG and GC, † equal to the rect-
 angle contained between DH and HE, to-
 gether with the square of BG; that is, the
 square of the perpendicular BG is less than
 the rectangle contained between AG and
 GC the segments of the axis, intercepted
 between its vertexes and the perpendicu-
 lar, by the rectangle contained between
 DH and HE, the segments of the axis, in-
 tercepted between the extremity of the
 line cut off it towards the center from the
 nearest vertex, equal to the line drawn
 from the given point to the nearest focus,
 and the foci D and E.

Therefore, *If from any point in the periphery of an ellipse, &c. which was to be demonstrated.*

Cor.

Cor. From the first part of this proposition it follows, that if any line, such as AC, be given, and bisected in F, and any other point, as E, between F and C be given, and from the point B BG be drawn perpendicular to AC and BE joined, to which let CH be made equal, cut off from C towards F; and if FC be to FE as FG is to FH, then will the point B be in the periphery of an ellipse, which has AC for its greater axis, and E for one of its foci.

For make AD equal to EC, and let an ellipse * be described, which has AC for its greater axis, and the points D and E for its foci, then will B be a point in the periphery of that ellipse; for if not, let the ellipse, if possible, pass through the point M, and join EM, and make CN equal to ME; then (by the above proposition) FN will be to FG as FE is to FC. But FE is to FC † as FH is to † *hypoth.* FG, therefore as FN is to FG ‡ so is ‡ *11 E. 1. 5.* FH to FG, and consequently FH is * * *2 E. 1. 5.* equal to FN, a part to the whole, which is absurd; therefore the ellipse does not cut the line BG in M; and after the same manner may it be demonstrated,

M. 2

that

that the ellipse cannot meet the line BG in any other point than B; therefore B is a point in the periphery of the ellipse.

PROPOSITION V.

THEOREM V.

If from any point in the periphery of an ellipse, a line be drawn parallel to one axis and cutting the other, the square of the axis to which it is drawn will be to the square of the axis to which it is parallel, as the rectangle contained between the segments of the axis, intercepted between the line drawn cutting it and its vertexes, is to the square of that line.

LET ABCD be any ellipse, AC the greater axis, BD the smaller, E and F the foci, and G the center, and H any point in the periphery; and from H let a line be drawn parallel to either axis, and first let it be parallel to the smaller axis BD, as HK, cutting the greater axis in K: I say, the square of the axis AC is to the square of the axis BD, as the rectangle
con-

contained between AK and KC is to the square of HK.

For join the lines HE and HF to the foci, and from the vertex C of the greater axis, nearest to the point H, cut off CL towards the center, equal to HF the smallest of the lines drawn from the point H to the foci.

Then because GC is to GF * as GK is * 4 of this.
to GL, the square of GC is † to the square † 22 E. l. 6,
of GF as the square of GK is to the square
of GL; and because the square of GC is
to the square of GF, as a part taken
from the one, *viz.* the square of GK,
is to a part taken from the other, *viz.*
the square of GL; therefore † the remain- † 19 E. l. 5;
der of the square GC, *viz.* * the rect- * 5 E. l. 2;
angle contained between AK and KC, is
to the remainder of the square of GF,
viz. the rectangle contained between EL
and LF, as the square of GC is to the
square of GF; and by † conversion, as † 19 E. l. 5;
the square of GC is to the rectangle con-
tained between AF and FC, so is the rect-
angle contained between AK and KC to
the excess by which the rectangle con-
tained between AK and KC exceeds the
rectangle contained between EL and LF;
that

* *4 of this.* that is, * the square of HK, and the rectangle contained between AF and FC, is
 † *2 of this.* † equal to the square of GB, half the smaller axis; therefore as the square of GC is to the square of GB, so is the rectangle contained between AK and KC to the square of HK; and by quadrupling the first ratio it will be, as the square of the axis AC is to the square of the axis BD, so is the rectangle contained between AK and KC, the segments of the axis, intercepted between its vertexes and the line, to the square of the cutting line KH.

Case 2. When the line drawn from the given point H, as HM, is parallel to the greater axis AC, and cutting the smaller axis BD in M; I say, the square of BD is to the square of AC, as the rectangle contained between BM and MD is to the square of MH.

For (by the above demonstration) as the square of GC is to the square of BG, so is the rectangle contained between AK and KC to the square of KH, or MG, which
 † *34 E. I. 1.* is † equal to it; by inversion it will be, as the square of BG is to the square of GC, so is the square of MG to the rectangle contained between AK and KC; and
 since

since the whole, *viz.* the square of BG, is to the whole, *viz.* the square of GC, as the square of MG, a part taken from the one, is to the rectangle contained between AK and KC, a part taken from the other, then will the * residue of the one, * 19 E. 1. 5: *viz.* † the rectangle contained between † 5 E. 1. 2: BM and MD, be to the residue of the other, *viz.* the square of GK, or MH which is equal to it, as the square of BG is to the square of GC; and by quadrupling this last ratio it is, as the square of the axis BD is to the square of the axis AC, so is the rectangle contained between BM and MD, the segments of the axis, intercepted between its vertexes and the cutting line, to the square of the line HM.

Therefore, *If from any point in the periphery of an ellipse, a line be drawn, cutting either of the axes, and parallel to the other, the square of the axis to which it is drawn is to the square of the axis to which it is parallel, as the rectangle contained between the segments of the axis, intercepted between the line drawn cutting it and its vertexes, is to the square of that line; which was to be demonstrated.*

Cor.

Cor. 1. From hence it follows, that if there be any number of points taken in the periphery of an ellipse, and from them lines be drawn parallel to one of the axes, and cutting the other, then will the squares of these parallel lines be to one another, as the rectangles contained between the segments of that axis to which they are drawn, and intercepted between the lines and its vertexes. Also right lines terminated both ways by the ellipse, and parallel to either axis, are bisected by the other; that is, the two axes are conjugate diameters. For let the lines HK, NO, be parallel to the axis BD, and meeting the axis AC in K and O; then (by the above prop.) it is, as the square of AC is to the square of BD, so is the rectangle contained between AK and KC to the square of KH, and so is the rectangle contained between AO and OC to the square of ON; and consequently as * the rectangle contained between AK and KC is to the rectangle contained between AO and OC, so is the square of KH to the square of ON. And if HK be produced until it cut the periphery in P,
HP

¶ 11 E. 1. 5.

HP will be bisected in K. For because the rectangle contained between AK and KC is to the square of KH, so is the same rectangle contained between AK and KC to the square of KP; therefore the square of HK is * equal to the square * 9 E. 1. 5 of KP, and consequently the line HK is equal to KP, that is, the two axes AC and BD are † conjugate diameters. † *def. 3 of this.*

Cor. 2. Any right line terminated both ways by the periphery of an ellipse, and bisected by either axis, is parallel to the other. Let the right line HP be terminated both ways by the periphery of the ellipse, and bisected in K by the axis AC, HP is parallel to the other axis BD; for if not, from H draw any other line, if possible, ‡ parallel to BD, ‡ 31 E. 1. 11 as HR, cutting the axis in T, and the periphery in R, and draw RS parallel to AC; which cuts HP in S; then because RS is parallel to TK, * as HT is * 2 E. 1. 6; to TR, so is HK to KS. But (by the above corol.) HT is equal to TR, therefore HK is equal to KS; but HK is † equal to KP, wherefore KS is equal to KP, a part to the whole, which is ab-

N

furd;

furd; therefore the right line HP is parallel to the axis BD.

Cor. 3. Equal right lines terminated by the periphery of an ellipse, and parallel to either of the axes, the segments of the other axis cut off by these lines to the center are equal; and if they be parallel to either of the axes, and the segments of the other axis cut off by them from the center be equal, they are equal to one another. For let HP and VY be equal to one another, and parallel to BD, GK will be equal to GX; because HP is equal to VY, HK is equal to VX; therefore XK is * equal and parallel to VH, and VM is † equal to MH, that is, XG is ‡ equal to GK; and after the same manner may it be demonstrated, that if GX be equal to GK, VX is equal to HK.

* *cor. 1. pr.*
5 of this.
 † 33 *E. l. 1.*
 ‡ 34 *E. l. 1.*

Cor. 4. An ordinate drawn to the transverse axis through either of the foci will be equal to the latus rectum of that axis; for through the focus F let ZFa be drawn an ordinate to the transverse axis AC; then because the square of GC is to the square of GB, as the rectangle contained between AF and FC, that is,

* the

* the square of GB, is to the square of * 2 of this.
 FZ, therefore as GC is to GB, so is
 GB to FZ; and by doubling the terms,
 as the transverse axis AC is to the con-
 jugate axis BD, so is the conjugate BD
 to the ordinate Za; therefore Za is † the ^{def. 11 of} _{this.}
 latus rectum of the transverse axis.

PROPOSITION VI.

THEOREM VI.

*If a circle be described, which has ei-
 ther of the axes of an ellipse for a
 diameter, and right lines drawn pa-
 rallel to the other axis, and produ-
 ced until they cut the periphery of
 the circle, the segments of these lines,
 intercepted between the given axis
 and the periphery of the circle, have
 the same proportion to one another,
 as the segments of the same lines in-
 tercepted between the given axis and
 the periphery of the ellipse.*

LET ABCD be any ellipse, AC the great-
 er, and BD the smaller axis, E the
 center, and let ALC be a circle which has
 the axis AC for a diameter, and from the
 N 2 points

points F and G in the ellipse let the right lines FH, GK, be drawn parallel to the axis BD, and cutting the periphery of the circle in L and M; I say, as FH is to GK, so is LH to MK.

For join the lines LA, MA, LC and MC;

Then because ALC is a semicircle, the

* 31 E. 1. 3. angle ALC is a * right angle, and LH is drawn perpendicular to the base AC; there-

† cor. 8. 1. E. fore as AH is † to LH, so is LH to HC; consequently the rectangle contained be-

‡ 16 E. 1. 6. tween AH and HC ‡ is equal to the square of LH; and for the same reason the rectangle contained between AK and KC is equal to the square of KM; therefore as the rectangle contained between AH and HC is to the rectangle contained between AK and KC, so is the square of HL to the square of KM. But as the rectangle con-

* cor. 1. pr. 5 of this. tained between AH and HC * is to the rectangle contained between AK and KC, so is the square of FH to the square of GK;

† 11 E. 1. 5. therefore as the square of LH † is to the square of MK, so is the square of FH to the square of GK; consequently as the line LH is to the line MK, so is the segment FH to the segment GK.

Therefore, *If a circle be described,*
which

which has either of the axes for a diameter, and right lines be drawn parallel to the other axis, and produced until they cut the periphery of the circle, the lines intercepted between the axis, which is the diameter of the circle, and its periphery, have the same proportion to one another, as the segments of the same lines intercepted between the axis and the periphery of the ellipse; which was to be demonstrated,

Cor. If LH be to MK as FH is to GK , and if the points F and G be in the periphery of the ellipse, and L in the periphery of the circle, the other point M will also be in the periphery of the circle; for if not, the circle will cut the line KM in some other point, which let be N : Then (by the above prop.) LH is to NK as FH is to GK . But * LH * *hypoth.* is to MK as FH is to GK , therefore as LH is to NK , † so is LH to MK ; consequently NK is ‡ equal to MK , a part † *11 E. l. 5.* to the whole, which is absurd. After the same manner may it be demonstrated, that if LH be to MK as FH is to GK , and the points L and M in the periphery of the circle, and F in the
pe-

periphery of an ellipse, the other point is also in the periphery of the ellipse.

L E M M A.

If two equal parts be cut from the extremities of any line, and any point taken between them, the rectangle contained between the segments of the given line, intercepted between its extremities and last assumed point, is equal to a rectangle contained between the segments of the line intercepted between its extremities and one of the first assumed points, together with the rectangle contained between the segments of the line intercepted between the last and two first assumed points (a).

LET AB be any right line, and from its extremities the two equal parts AC, BD, are cut off, and E any point assumed betwixt C and D; I say, the rectangle contained between AE and EB is equal to the rectangle contained between AD and DB, together with the rectangle contained between CE and ED,

For

(a) This lemma is taken from the 178 prop. of the 7th book of Pappus Alexandrinus, collect. mathemat.

For * bisect the line AB in F. * 10 E. 1. 1.

Then because the line AB is equally cut in F, and unequally cut in E, the † rect- † 5 E. 1. 2.
angle contained between AE and EB, together with the square of FE, is equal to the square of FB; for the same reason the rectangle contained between AD and DB, together with the square of FD, is equal to the square of FB. Again, because AF is equal to FB, and AC to DB, therefore CF is equal to FD; and consequently the † rectangle contained between CE and ED, † 5 E. 1. 2.
together with the square of FE, is equal to the square of FD; therefore the rectangle contained between AD and DB, together with the rectangle contained between CE and ED, together with the square of FE, is equal to the square of FB; that is, to the rectangle contained between AE and EB, together with the square of FE; take the square of FE which is common from both, and there will remain the rectangle contained between AE and EB, the segments of the line intercepted between its extremities and last assumed point, equal to the rectangle contained between AD and DB, the segments of the line intercepted between its extremities and one
of

of the first assumed points, together with the rectangle contained between CE and ED, the segments of the line intercepted between the last point and the two first assumed points; *which was to be demonstrated.*

PROPOSITION VII.

THEOREM VII.

In an ellipse the transverse axis is the greatest diameter, and the smaller axis the least, and among the other diameters those that are nearer the transverse axis are greater than those more remote.

LET ABCD be any ellipse, whose transverse axis is AC, and smaller axis BD, E the center, and FG and HK any other diameters; I say, AC the transverse axis is the greatest diameter in the ellipse, BD the smaller axis is the least, and FG nearer the transverse axis is greater than HK more remote.

For from the point F draw the line FL
* 12 E. 1. 1. perpendicular to AC, and FO perpendicular to BD.

Then

Then because the line FL is drawn perpendicular to the transverse axis, the rectangle contained between AL and LC is * greater than the square of FL; add to * *4 of this* both the square of EL, which is common, then will the rectangle contained between AL and LC, together with the square of EL, be greater than the two squares described upon EL and LF. But the rectangle contained between AL and LC, together with the square of EL, † is equal † *5 E. 1. 2* to the square of EC, and the two squares described upon EL and LF are ‡ equal to ‡ *47 E. 1. 1* the square of EF; therefore the square of EC is greater than the square of EF, and consequently the line EC greater than EF, and its double, *viz.* the transverse axis AC greater than the double of EF, *viz.* the diameter FG. After the same manner is the transverse axis proven greater than any other diameter; therefore *the transverse axis is the greatest diameter drawn within an ellipse.*

Again, because the line FO is drawn perpendicular to the axis BD, the square of BD is * to the square of AC as the rectangle contained between DO and OB is to the square of OF; but the square of BD

is smaller than the square of AC, therefore the rectangle contained between DO and OB is * smaller than the square of OF; add the common square of OE to both, then is the rectangle contained between DO and OB, together with the square of OE, smaller than the two squares described upon OE and OF. But the rectangle contained between DO and OB, together † 5 E. l. 2. with the square of OE, is † equal to the square of EB, and the two squares described upon OE and OF are ‡ 47 E. l. 1. ‡ equal to the square of EF; consequently the line EB is smaller than EF; wherefore the smaller axis BD is smaller than the diameter GF; and after the same manner may it be demonstrated, that *the smaller axis is less than any other diameter in the ellipse.*

I say likewise, the diameter FG nearer the transverse axis is greater than KH more remote:

¶ 11 E. l. 1. For from H draw HM* perpendicular to AC the transverse axis, which produce until it cut the periphery in Q, and from the vertex A cut a part off it, as AN, equal to CL; Then because from the points H and F in the ellipse the perpendiculars HM and FL are drawn to the axis AC, as the rect-

rectangle contained between AM and MC is to the square of HM, * so is the rectangle contained between AL and LC to the square of FL, or PM which is † equal † 34 E. l. 2. to it. But the rectangle contained between AM and MC † exceeds the rectangle contained between AL and LC, by the rectangle contained between NM and ML; and because HQ is * bisected in M, the square of HM exceeds the square of MP, † by the rectangle contained between HP and PQ; † 5 E. l. 2. therefore the residue, *viz.* the rectangle contained between NM and ML, † is to the residue, *viz.* the rectangle contained between HP and PQ, as the whole rectangle contained between AM and MC is to the whole, *viz.* the square of HM. But the rectangle contained between AM and MC is * greater than the square of HM, therefore the rectangle contained between NM and ML is † greater than the rectangle contained between HP and PQ; add to both the square of EM, then the rectangle contained between NM and ML, together with the square of EM, that is, the † square of EL, is greater than the rectangle contained between HP and PQ, together with the square of EM. Again, add

to both the equal squares of PM and FL, then will the rectangle contained between HP and PQ, together with the square of PM, that is, the square of HM, together with the square of ME, be smaller than the two squares described upon FL and EL. But the two squares described upon HM and ME are * equal to the square of EH, and the two squares of EL and LF are equal to the square of EF; therefore the square of EF is greater than the square of EH, and the line EF greater than the line EH; and consequently the diameter FG nearer the transverse axis is greater than the diameter AK more remote.

Therefore, *In an ellipse the transverse axis is the greatest diameter, and the smaller axis the least; and among the other diameters those that are nearer the transverse axis are greater than those more remote; which was to be demonstrated.*

PROPOSITION VIII.

THEOREM VIII.

If from any point in the periphery of an ellipse, which is not the vertex of either

either axis, a right line be drawn parallel to one axis, and produced until it cut the periphery of a circle described, having the other axis for a diameter, and if from that point a tangent be drawn to the circle, that tangent produced will cut the line which is both the diameter of the circle and the other axis of the ellipse; and if from that point of intersection a line be drawn to the given point in the periphery of the ellipse, that line is a tangent to the ellipse: Or if a line be drawn a tangent to the ellipse, and from the point of intersection of the tangent and common diameter of the circle and ellipse, a line be drawn to the point in the periphery of the circle, where a line drawn from the point of the ellipse parallel to the other axis cuts it, that line is a tangent to the circle; also a line drawn from the vertex of one axis parallel to the other is a tangent to the ellipse.

LET ABCD be an ellipse, AC and BD its two axes, and E its center, and
let

let the circle AFC be described upon AC, and from the point H in the periphery of the ellipse, which is not the vertex of either axis, let the right line HK be drawn parallel to the axis BD, and cutting the periphery of the circle in F, and from F draw FG a tangent to the circle, FG produced will meet the axis AC: For join FE, then because FG is a tangent, the angle
 118 E.L. 3. GFE is a * right angle, and BEC is a right angle, consequently the two angles GFE and FEC are less than two right angles; therefore the lines FG and EC produced will meet; let them meet in the point L, and join LH; I say, LH is a tangent to the ellipse;

For if it be not, let it, if possible, meet the periphery of the ellipse in any other
 119 E.L. 1. point, as M, and from M draw MN † parallel to the axis BD, which produced cuts the line FL in O.

Then because MN is parallel to HK,
 120 E.L. 1. the triangles MNL and HKL are ‡ equiangular; and for the same reason the triangles FKL and ONL are equiangular;
 * 4 E.L. 6. therefore as LN is to LK, * so is NM to KH. Again, as LN is to LK, so is NQ to KF; consequently as NM is to KH,
 * so

Book II. *Elements of Conic Sections.* 111

* so is NO to KF. But H and M are ^{*11 E.L.31} points in the periphery of the ellipse, and F is a point in the periphery of the circle, therefore the other point O is likewise [†] in the periphery of the ~~ellipse~~ ^{† cor. pr. 4 of this.}; wherefore the line FL, which is a tangent to the circle, touches it in more than one point, which is absurd; consequently the line HL is a tangent to the ellipse, W.W.D. + circle

After the same manner may it be demonstrated, that if HL is a tangent to the ellipse, FL is also a tangent to the circle; also if through B, the vertex of the axis BD, a right line be drawn parallel to the other axis AC, that line is a tangent to the ellipse.

For if not, let it meet the ellipse in another point, as P, and through P draw PQ [†] parallel to BD, intersecting AC in ^{†31 E.L.11} Q, and the circle in S, and let EB produced cut the circle in R, and join RS.

Then because PB is parallel to QE, and BE to PQ, PE is a parallelogram, and PQ is * equal to BE; but as BE is to PQ, ^{*34 E.L.11} [†] so is ER to ~~EB~~; therefore RE is equal ^{† 6 of this.} to SQ, and parallel to it, consequently RS is parallel and equal to QE, and the angle SRE is a right angle, and from R the

the extremity of the diameter of a circle, is drawn a right line at right angles to the diameter falling within the circle, * which is absurd; therefore the line BP does not touch the periphery of the ellipse in any other point than B, and consequently is a tangent to it.

* cor. 16.
E. I. 3.

Therefore, *If from any point in the periphery of an ellipse, &c. which was to be demonstrated.*

Cor. 1. From hence it follows, that from one point no more than one line can be drawn a tangent to the ellipse; for if two lines could be tangents to the ellipse in the same point, then two right lines would also be tangents to a circle, and touching in the same point, † which is absurd.

† 16E. I. 3.

Cor. 2. Any right line cutting an ellipse, cuts it in no more than two points; for if it cut it in more points, then would a line cut the circle also in more points than two, which is absurd.

Cor. 3. The angle comprehended between any diameter, except the axis, and a tangent drawn through its vertex towards that side, which produced cut the axis, is greater than a right angle.

Let

Let HE be a diameter, HL a tangent, and AC the greater axis, (the rest of the construction remaining the same as in the above prop.) then will the angle EHL be * greater than the angle EFL, * $\angle EHL$ which is a right angle.

Cor. 4. From hence it is evident how to draw a tangent to an ellipse, from a given point in it if the greater axis be given.

PROPOSITION IX.

THEOREM IX.

Right lines drawn tangents to an ellipse, from the vertexes of any diameter, are parallel to one another.

LET ABCD be any ellipse, AC its greater, and BD its smaller axis, and HT any diameter, and from its vertexes H and T let HL and TX be drawn tangents to the ellipse; I say, HL is parallel to TX.

For let the tangents produced meet the axis AC in the points L and X, and describe a circle which has AC for a diameter, and from H and T draw the lines

P

PHK

114 *Elements of Conic Sections. Book II.*

*₃₁ *E. l. 1.* HK and TY * parallel to the axis BD, and cutting the periphery of the circle in F and V, and join the lines FL, VX, FE and YE.

Then because HL and TX are tangents
†₃ *of this.* to the ellipse, FL and VX will be † tangents to the circle; and the triangles EHK and ETY having the angles HKE and
†₁₅ *E. l. 1.* TYE right angles, and YET † equal to
*₃ *of this.* HEK, and the side ET * equal to the side
†₄ *E. l. 1.* EH, therefore HK is † equal to YT. But
†₆ *of this.* as KH is to TY, † so is KF to YV; wherefore the triangles FEK and YEV are equal in all respects, and the angle FEK equal to the angle YEV; add to both the common angle VEK, then will the two angles VEK and VEY be equal to the two angles VEK and KEF. But the two angles
†₁₃ *E. l. 1.* VEK and VEY are * equal to two right angles; wherefore the two angles VEK and KEF are equal to two right angles, and consequently the two lines FE and
†₁₄ *E. l. 1.* EV † make up one right line. Again, because KF is equal to VY, and the angles FKL and XYV both right angles, also the angles KFL and YVX equal to each other, being the complements of the two equal angles EFK and EVY to a right angle; where-

Book II. *Elements of Conic Sections.* 115

wherefore the side KL is * equal to the ^{*26 E. I. I.} side YX, and KH is equal to YT, and the contained angles right; therefore the angle HLK is † equal to the angle YXT; ^{† 4 E. I. I.} and consequently the tangent HL is ‡ pa- ^{‡ 26 E. I. I.} rallel to the tangent TX.

Therefore, *Right lines drawn tangents to an ellipse, from the vertexes of any diameter, are parallel to one another; which was to be demonstrated.*

PROPOSITION X.

THEOREM X.

If from any point in the periphery of an ellipse two right lines be drawn to the foci, and from the same point a line be drawn bisecting the outward angle contained between one of these lines and the other produced, that line is a tangent to the ellipse; and if from any point in the periphery of an ellipse a tangent be drawn, it will bisect the outward angle contained between one of the lines drawn from the point of contact to the foci and the other produced.

P 2

LET

LET ABC be any ellipse, AC the greater axis, H the center, and D and E the foci, and from any point, as B, in the periphery, the right lines BD and BE be drawn to the foci, one of which, as DB, is produced to G, and from B the right line BF is drawn, bisecting the angle EBG; I say, the line BF is a tangent to the ellipse.

Case 1. When the line BF is parallel to the greater axis.

Make BG equal to BE, and join GE which cuts BF in F.

Then because BG is equal to BE, and BF is common, and the contained angle GBF is * equal to the contained angle EBF, † *E. 1. 1.* therefore the base GF is † equal to the base FE. Again, because BF is parallel to DE, ‡ *E. 1. 6.* as GF is to FE, † so is GB to BD; therefore GB is equal to BD; but BG is equal to BE, therefore BD is equal to BE, and BH is common, and DH is equal to HE; consequently the triangles DBH and BHE have the three sides of the one equal to the sides of the other; therefore the angle BHD is * equal to the angle BHE, and consequently both are right angles; wherefore BH is the † smaller axis; and since

BF

BF is drawn from the vertex of the axis BH parallel to the other axis, * the line BF is a tangent to the ellipse; W. W. D. ^{9 of this}

Case 2. When the line BF is not parallel to the axis AC.

Produce DB, and make BG equal to BE, and join GE cutting the line BF in F, and take any point in the line BF, as K, and join the lines KD, KE and KG.

Then because BG is equal to BE, and BF is common, and the contained angle GBF equal to the contained angle FBE, therefore the base GF is † equal to the † base FE, and the angles GFB and BFE are equal to one another, and consequently right angles. Again, because GF is equal to FE, and FK is common, and the contained angles GFK, KFE, are each right angles, therefore the base KE is equal to the base KG. But DK and KG taken together are † greater than DG, that is, greater than DB and BE; but DB and BE are * equal to the greater axis AC, therefore † DK and KG, that is, DK and KE taken together, are greater than the greater axis AC; consequently the point K is † without the ellipse. After the same manner may it be demonstrated, that any other point ^{† 20 E. 1. 1.} ^{* 1 of this} ^{† cor. 3. pr. 1 of this}

point taken in the line BF will be without the ellipse, therefore the line BF * is a tangent to the ellipse.

* *def. 12 of this.*

Lastly, If BF be a tangent to the ellipse, and from B the point of contact two right lines, as BD, BE, be drawn to the foci, one of which being produced to G; I say, the angle GBF is equal to the angle FBE; for if they are not equal to one another, then from B another line may be drawn bisecting the angle GBE, which (by the preceding demonstration) is a tangent to the ellipse. But BF is a † tangent to the ellipse, and consequently from the same point B are drawn more than one line a tangent to the ellipse, ‡ which is absurd; wherefore the angle GBF is equal to the angle FBE.

† *hypothesis.*

‡ *cor. 1. pr. 8 of this.*

Therefore, *If from any point in the periphery of an ellipse two right lines be drawn to the foci, and from the same point a line be drawn bisecting the outward angle contained between one of these lines and the other produced, that line is a tangent to the ellipse; and if from any point in the periphery of an ellipse a tangent be drawn, it will bisect the outward angle contained between one of these lines drawn from the point*

point of contact to the foci and the other produced; which was to be demonstrated.

Cor. From hence it follows, that the angles contained between the tangent and the right lines drawn from the point of contact to the foci are equal to one another; that is, the angle DBK is equal to the angle EBF, for if not, then will the two angles EBF, FBG, be also unequal, which is (by this) demonstrated to be absurd.

Also from this proposition it is evident how to draw a tangent to an ellipse from a given point in its periphery, the foci being given.

PROPOSITION XI.

PROBLEM I.

The foci and greater axis of any ellipse being given, to draw a right line a tangent to the ellipse, and parallel to another right line given.

LET AC be the greater axis, D and E the foci of any ellipse, and MN a right line given; it is required to draw a tangent to the given ellipse parallel to the given line MN. From

From either of the foci, as E, draw EL
212 E. 1. 1. * perpendicular to MN, and about the o-
 ther focus D as a center, with a distance
 equal to AC, describe a circle which cuts
 the line EL in two points; let one of these
 points be G, join DG, and with the line
213 L. 1. 1. GE, and at the point E, † make the angle
 GEB equal to the angle EGB, and through
 B the point of intersection of these two
214 E. 1. 1. lines draw BF ‡ parallel to MN.

Then because the angles BGE and BEG
 are equal to one another, the line BG is
216 E. 1. 1. * equal to BE, wherefore the two lines DB
 and BE taken together are equal to DG,
 that is, to the greater axis AC; conse-
217 Cor. 3. Pr
 2 of this. quently the point B is ‡ in the periphery
 of the ellipse. Again, let BF cut the line
 GE in the point F; then because GE is at
 right angles to MN, and BF parallel to it,
 therefore the angles BFG, BFE, are right
 angles; and the angles BEF, BGF, are e-
 qual to one another, consequently the re-
 maining angles EBF and FBG are also e-
218 of this. qual to one another; therefore BF is a ‡
 tangent to the ellipse, and parallel to the
 given line MN; *which was to be done.*

After the same manner may another line
 be drawn a tangent to the ellipse, and pa-
 rallel

rallel to the given line, by means of that other point where the circle described about the center D cuts the line EL.

PROPOSITION XII.

THEOREM XI.

Any right line terminated both ways by the periphery of an ellipse, and parallel to a tangent, is bisected by a diameter drawn through the point of contact; and if it be bisected by any diameter, it will be parallel to a tangent drawn from the vertex of that diameter. Also two diameters, one of which is parallel to a tangent drawn from the vertex of the other, are conjugate diameters.

FIRST, if the diameter that is drawn through the point of contact be either of the axes, the tangent drawn through its vertex will be * parallel to the other * ^{of this} axis, and consequently the right line itself will be parallel to that other axis, and therefore † bisected by the axis passing ^{cor. 1. pr} through the point of contact; but if the ^{of this}

Q

dia

diameter that passes through the point of contact is any other diameter,

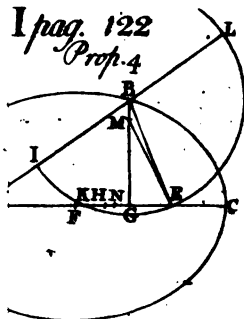
Let ABC be any ellipse, AC its greater axis, D its center, EF a right line terminated both ways by the ellipse, and parallel to the tangent BH, and let the diameter BD be drawn cutting the line EF in G; I say, the line EF is bisected in the point G.

For let the lines EF and BH produced cut the diameter AC in H and K; and about D as a center, with the distance DA, describe the circle ALC; and from the points E, B and F, draw the lines EM, BN and FO, * parallel to RD the other axis, and which produced cut the periphery of the circle in P, L and Q; join the lines LH, LD, PK, which last line cuts LD in T, the line PK will pass through the point Q, for if not, it will cut the line QO in some other point than Q, as in S.

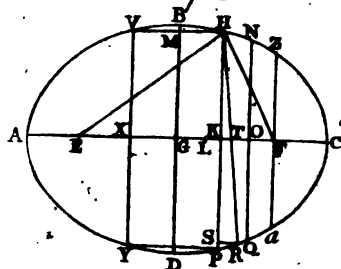
Then because PM is parallel to SO, the
 [29 E.L. 1. triangles PMK, SOK, are † equiangular; and therefore as PM is to SO, so is MK to OK; for the same reason the triangles EMK, FOK, are similar, and as EM is to FO, so is MK to OK, wherefore as EM
 [11 E.L. 1. is to FO, so ‡ is MP to OS: But as EM
 is

I pag. 122

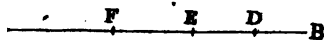
Prop. 4



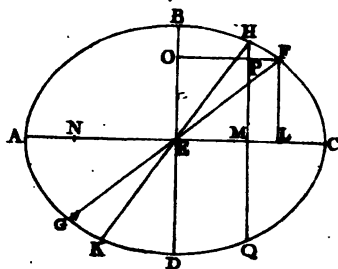
Prop. 5



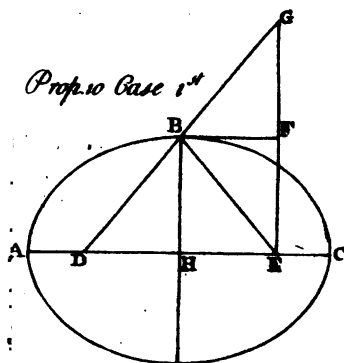
Lemma



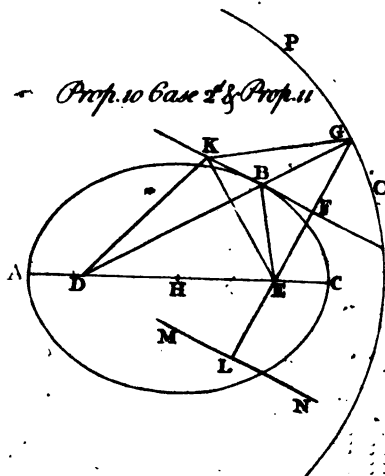
Prop. 7

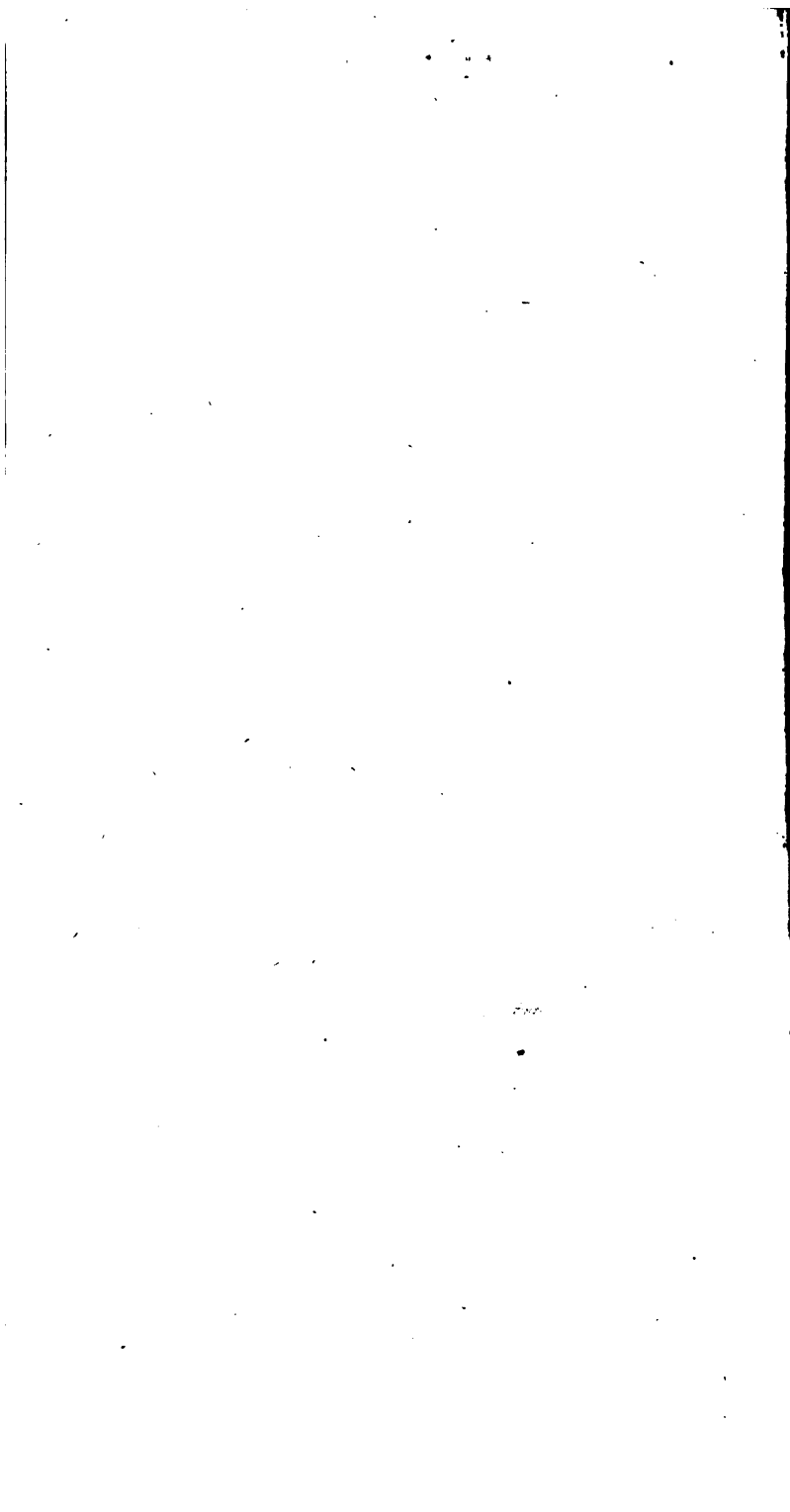


Prop. 10 Case 1st



Prop. 10 Case 2nd & Prop. 11





Book II. *Elements of Conic Sections.* 123

is to FO, * so is PM to QO, and conse- * 6 of this.
 quently SO is † equal to QO, a part to † 9 E. 1. 2
 the whole, which is absurd. After the
 same manner may it be demonstrated, that
 PK cannot cut the line QO in any other
 point than Q. Again, because BN is pa-
 rallel to FO, and GK to BH, the triangles
 FOK, BNH, are equiangular; therefore
 as BN is to FO, † so is NH to OK; but † 4 E. 1. 6.
 as BN is to FO, † so is LN to QO; there- † 6 of this
 fore as LN is to QO, so is NH to OK;
 and the angles LNH, QOK, are right angles;
 wherefore the triangles LNH, QOK, are
 * similar, and the angles LHN, QKO, are * 6 E. 1. 6;
 equal, consequently the lines PK and LH
 are parallel; also because GK is parallel to
 BH, as ~~is~~ is to KH, † so is DG to GB; † 2 E. 1. 6; † DK
 also because KT is proven parallel to LH,
 as DK is to KH, so is DT to TL; there-
 fore as DG is to GB, so is DT to TL,
 wherefore TG is † parallel to LB. Again, † 2 E. 1. 6:
 because BH is a tangent to the ellipse, there-
 fore LH is a * tangent to the circle, con- * 8 of this.
 sequently the angle HLD is a † right angle; † 29 E. 1. 1.
 but because PQ is parallel to LH, the angle
 DTQ is equal to the angle DLH; there-
 fore the angle DTQ is a right angle, con-
 sequently the line TP is † equal to the † 3 E. 1. 1

line TQ; and because the lines PE, TG,
^{E. 1. 6.} QF, are all parallel, as QT is to TP, * so
 is FG to GE. But QT is equal to TP,
 therefore FG is equal to GE, consequent-
 ly EF is bisected in G.

Again if EF be bisected in G by the di-
 ameter BD, a tangent drawn through the
 vertex B will be parallel to the line EF;
 for if not, some other tangent may be drawn
^{of this.} to the ellipse † parallel to EF, and (by the
 above demonst.) will be bisected by a di-
 ameter drawn through the point of contact;
^{hypothes.} but it is † bisected by the diameter BD,
 therefore the line EF will be bisected by
 two different lines cutting it in different
 points, which is absurd.

Also if BE and GH be two diameters,
 and BF a tangent to the ellipse drawn from
 the vertex of the diameter BE, and if GH
 the other diameter be parallel to the tan-
 gent BF; I say, the two diameters BE and
 GH are conjugate diameters: For let AC
 be either of the axes, and D the center of
 the ellipse, and the points P and Q its fo-
 ci, and upon AC, as a diameter, describe
 the circle AKN, and from the points B
^{E. 1. 1.} and H draw the lines BL and HM * pa-
 rallel to the other axis, and cutting the pe-
 riphe-

riphery of the circle in K and N, and from H draw HO a * tangent to the ellipse, and let the tangents BF and HO produced cut the axis in the points F and O, and join the lines KF, ON, KD and ND. * cor. pr. 10 of this.

Then because BF and HO are tangents to the ellipse, cutting the axis AC in the points F and O, and from these points the lines FK and ON are drawn to these points of the periphery of the circle, where perpendiculars to the axis drawn through the points of contact cut it, these lines FK and ON are † tangents to the circle; and because the line BL is parallel to HM, and DH † parallel to BF, therefore the triangles BLF and DHM are equiangular; consequently as BL is to HM, * so is LF to DM; but as BL is to HM, † so is KL to NM; therefore as KL is to NM, so is LF to DM; and the angles KLF, DMN, are right angles, consequently the triangles DMN, KLF, are † similar, and the angle KFL equal to the angle MDN; therefore the line KF is * parallel to DN. Again, * because KF is a tangent to the circle, and KD is drawn to the center, the angle FKD is a † right angle; and for the same reason the angle OND is a right angle; and since

126 *Elements of Conic Sections. Book II.*

since DN is parallel to KF, the angle KDN
 * 29 E. l. 1. is a * right angle, wherefore the line DK
 † 28 E. l. 1. is † parallel to NO, and the angle KDO
 equal to the angle DON. But the angles
 KLD, NMO, are right angles, therefore
 ‡ 4 E. l. 6. as † KL is to MN, so is LD to MO; but
 * 6 of this. as KL is to MN, * so is BL to HM; there-
 † 6 E. l. 6. fore the triangles BLD and HMO are † si-
 milar, and the angles BDL, MOH, are e-
 qual to one another, consequently the di-
 ameter BD is parallel to the tangent HO.
 But right lines terminated by the periphe-
 ry of the ellipse, and parallel to HO, that
 is, to the diameter, are (by the above de-
 monstration) bisected by the diameter HD;
 and for the same reason the diameter BD
 bisects all the lines terminated both ways
 by the periphery of the ellipse, parallel to
 the diameter HD; therefore the two dia-
 ‡ def. 3 of this. meters BE and HG are † conjugate dia-
 meters.

Therefore, *Any right line terminated
 both ways by the periphery of an ellipse,
 and parallel to a tangent, is bisected by
 a diameter drawn through the point of
 contact; and if it be bisected by any di-
 ameter, it will be parallel to a tangent
 drawn from the vertex of that diame-
 ter.*

ter. Also two diameters, one of which is parallel to a tangent drawn from the vertex of the other, are conjugate diameters; which was to be demonstrated.

Cor. 1. All the ordinates to the same diameter are parallel to one another. Also, if two or more right lines be terminated both ways by the periphery of an ellipse, the diameter which bisects one of them will also bisect all the rest, because the line which is bisected is * parallel to a tangent drawn through the vertex of the diameter which bisects it, consequently they will be † parallel to the same tangent; therefore they are all ‡ bisected by the same diameters.

Cor. 2. The line which bisects two parallel right lines, terminated both ways by the periphery of the ellipse, is a diameter; for if not, any diameter which bisects one of them will (by the preced. cor.) bisect the other, and those two lines will be bisected by two different lines, which is absurd.

Cor. 3. A right line drawn from the vertex of any diameter, parallel to an ordinate to the same diameter, is a tangent to the ellipse.

Cor.

* part 2.
prec. prop.

Cor. 4. Two right lines cutting one another, and terminated both ways by the periphery of an ellipse, and not drawn through the center, will not mutually bisect each other; for if they do bisect each other, then are they both * parallel to a tangent drawn from the vertex of that diameter, which is drawn through the point of intersection of these right lines; and consequently both these lines are parallel to one another, which is absurd.

Cor. 5. Conjugate diameters are each parallel to a tangent drawn through the vertex of the other diameter. Also one diameter can have no more than one diameter for its conjugate; for if it had another, then would they both be parallel to a tangent drawn through the vertex of the first given diameter, and consequently parallel to one another, which is absurd.

Cor. 6. A right line drawn from the vertex of any diameter, parallel to its conjugate diameter, is a tangent to the ellipse.

Cor. 7. A right line parallel to any diameter, and terminated both ways by the

the periphery of the ellipse, is bisected by its conjugate diameter, because it will be parallel to a tangent drawn through the vertex of that conjugate, and therefore will be * bisected by the same diameter; and if it be bisected by any diameter, it will be parallel to its conjugate diameter.

* *part of
prec. prop.*

Cor. 8. If a tangent be drawn to a circle, which has either of the axes of an ellipse for a diameter, and from the point of contact a line be drawn perpendicular to the axis, and through the extremities of any ordinate to that diameter, which is drawn through the intersection of the periphery of the ellipse, and the line drawn perpendicular to the axis, two right lines be drawn perpendicular to the same axis, and cutting the periphery of the circle, and if these points of intersection be joined with a right line, that line is parallel to the tangent of the circle.

R

P R O

PROPOSITION XIII.

THEOREM XII.

If from any point in the periphery of an ellipse a right line be drawn cutting any diameter, and parallel to its conjugate, the square of the diameter to which it is drawn will be to the square of the diameter to which it is parallel, as the rectangle contained between the segments of the first diameter, intercepted between its vertexes and the line that cuts it, is to the square of the line drawn cutting it; and if the square of the diameter to which the line is drawn, be to the square of the diameter to which it is parallel, as the rectangle contained between the segments of the first mentioned diameter, intercepted between the line drawn and its vertexes, is to the square of the line drawn cutting it, the point from which the line is drawn is a point in the periphery of the ellipse.

LET ABCD be any ellipse, AC and BD two conjugate diameters, and EF a
right

right line drawn parallel to the diameter DB, and cutting AC in F; I say, the square of AC, the diameter to which it is drawn, is to the square of the diameter BD, to which it is parallel, as the rectangle contained between AF and FC, the segments of the diameter intercepted between the line drawn and its vertexes, is to the square of EF the line drawn cutting it.

For let GH be either of the axes, and I the center, and about the center I, with the distance IG or IH, describe a circle, and let EF produced cut the periphery of the ellipse in K, and from the points A, E, B and K, draw the lines AL, EM, BN and KO, * perpendicular to the axis GH, * ^{12 E. l. r.} and cutting the periphery of the circle in the points P, Q, R and S; and from the points P and R draw the lines PT and RV † tangents to the circle, and cutting † ^{16 E. l. 3.} the axis GH produced in the points T and V; join the lines TA, VB, which will be † tangents to the ellipse; produce EF which † ^{8 of this.} is parallel to AT, until it cut the axis in X; then will the points X, Q and S be in a * right line that will be † parallel to * ^{12 of this.} PT; join the lines PI, IR and YF, which † ^{cor. 8. pr. 12 of this.} last produce until it cut the axis GH in Z,

132 *Elements of Conic Sections. Book II,*

and through the point E draw the line
 *31 E. I. 1. EW * parallel to GH, and cutting YZ in
 W.

Then because AT is parallel to FX, as
 †2 E. I. 6 IX is to XT † so is IF to FA; and for
 the same reason as IX is to XT so is IY
 to YP; consequently the line YF will be
 parallel to PL; and because IR (by the
 preceding prop.) was proved parallel to
 PT, the angle PIR is a right angle; but
 the two angles LIP, IPL, are equal to a
 right angle between them; consequently
 the angles LIP, IPL, are equal to the angle
 PIR; take from both the common angle
 PIL, and there will remain the angle IPL
 equal to the angle NIR; and the angles
 ILP, INR, are right angles, and the side
 IP equal to the side IR, therefore IN is
 †26 E. I. 1. † equal to PL. Again, because the angle
 IPT is a right angle, and PL is drawn per-
 pendicular to IT; therefore the triangle
 †8 E. I. 6. IPL is * similar to the triangle LPT; and
 because the line YZ is parallel to PL, and
 YX to PT, the triangles LPT, XYZ, are
 equiangular; therefore the triangle IPL is
 equiangular to the triangle XYZ; where-
 †4 E. I. 6. fore as IP is to PL, † so is XY to XZ.
 But because QM is parallel to YZ, as XY

is

is to XZ, * so is QY to MZ; therefore * 2 E. 1. 6;
as IP is to PL, so is QY to MZ. But
LP is equal to IN, and MZ is equal to
EW; wherefore as IP is to IN, so is QY
to EW; and (by alternation) as IP is to
QY, so is IN to EW. But because IN is
parallel to EW, and IB to EF, as IN is to
EW, † so is IB to EF; therefore as IP is † 4 E. 1. 6;
to QY, † so is IB to EF; consequently as † 11 E. 1. 5;
the square of IP is to the square of QY,
* so is the square of IB to the square of EF. * 22 E. 1. 6;
But because the four lines IP, IY, IA, IF,
are proportionals, it will be as the square
of IP is to the square of IY, so is the
square of IA to the square of IF; and as
the square of IP is to the excess by which
the square of IP exceeds the square of IY,
that is, the square of YQ, so is the square
of IA to the excess by which the square
of IA exceeds the square of IF, that is, † † 5 E. 1. 2;
the rectangle contained between CF and
FA. But before it was as the square of IP
is to the square of QY, so is the square of
IB to the square of EF; therefore as the
square of IA is to the rectangle contained
between CF and FA, so is the square of
IB to the square of EF; and by quadru-
pling the antecedents, it will be as the
square

square of AC is to the rectangle contained between CF and FA, so is the square of DB to the square of EF; and by alternation, as the square of AC, the diameter to which the line is drawn, is to the square of BD the diameter to which it is parallel, so is the rectangle contained between CF and FA the segments of the diameter, intercepted between the point where the line cuts it and its vertexes, to the square of EF the line drawn cutting it; W. W. D.

Again, if from any point E, EF be drawn parallel to the diameter BD, and cutting its conjugate diameter AC in F, and if the square of AC be to the square of BD as the rectangle contained between AF and FC is to the square of EF; I say, the point E is a point in the periphery of the ellipse; for if the periphery of the ellipse does not cut the line EF in the point E, it will cut it in some other point, which, if possible, let be in e; then because e is a point in the periphery of the ellipse, eF is drawn parallel to the diameter BD, and cutting its conjugate diameter AC; (by the above demonstration) it will be as the square of AC is to the square of BD, so is the rectangle contained between CF and FA to the square of
of

of Fe. But as the square of AC is to the square of BD, * so is the rectangle contained between AF and FC to the square of EF; therefore as the rectangle contained between AF and FC is to the square of EF, † so is the rectangle contained between AF and FC to the square of eF; consequently the square of EF is ‡ equal to the square of eF, and the line EF equal to the line eF, a part to the whole, which is absurd. After the same manner may it be demonstrated, that the periphery of the ellipse cannot cut the line EF in any other Point than E.

Therefore, *If from any point in the periphery of an ellipse, &c. which was to be demonstrated.*

Cor. 1. From hence it follows, that because the ordinates of any diameter are * parallel to its conjugate diameter, EF ^{* 12 of this} may therefore be considered as a semi-ordinate to the diameter AC; and consequently the square of every semi-ordinate to the same diameter are to one another, as the rectangles contained between the segments of the diameter, intercepted between the vertexes, and the respective ordinates.

Cor.

Cor. 2. If from any two points, E and a, one of which, as E, is in the periphery of the ellipse, the two lines EF and ab be drawn parallel to the ordinates of the diameter AC, and cutting it in F and b; and if the square of EF be to the square of ab as the rectangle contained between AF and FC is to the rectangle contained between Ab and bC, the other point a will be a point in the periphery of the ellipse; for if the periphery of the ellipse does not cut the line ab in a, let it cut it in any other point, as in d, then will db and EF be two semi-ordinates; therefore as the square of EF is to the square of db, * so is the rectangle contained between AF and FC to the rectangle contained between Ab and bC; wherefore as the rectangle contained between Ab and bC is to the square of ab, † so is the rectangle contained between Ab and bC to the square of bd; and ‡ consequently the square of ba is ‡ equal to the square of bd, and the line ba equal to the line bd, a part to the whole, which is absurd; therefore the point a is in the periphery of the ellipse.

Cor. 3. Any diameter is to its latus rectum

as

* *prec. cor.*

† *E. I. 5.*

‡ *E. I. 5.*

as a rectangle contained between the segments of that diameter, intercepted between its vertexes and an ordinate, is to the square of that ordinate: For let gh be the latus rectum of the diameter AC; then because AC is to its conjugate BD, * as BD is to gh; therefore AC is to gh † as the square of AC is to the square BD. But as the square of AC is to the square of BD, ‡ so is the rectangle contained between AF and FC to the square of FE; therefore as the diameter AC is to its latus rectum gh, * so is the rectangle contained between AF and FC to the square of FE.

* def. 11 of this.
† cor. 20 E. l. 6.
‡ 13 of this;

PROPOSITION XIV.

THEOREM XIII.

If a circle be described having for its diameter any diameter of an ellipse, and two or more semi-ordinates be drawn to that diameter, and if from the points of intersection of the semi-ordinates and diameter perpendiculars be drawn to that diameter, and cutting the periphery of the circle, the semi-ordinates will have the same

S

pro-

proportion to one another as the segments of the perpendiculars intercepted between the diameter and periphery of the circle; and if the semi-ordinates have the same proportion to one another as perpendiculars to the diameter, drawn from the point of intersection of the semi-ordinates and diameter, and if the extremity of one of those lines be in the periphery of a circle described upon that diameter, the extremity of the other line will also be in the periphery of the circle; or if the perpendiculars from the periphery of the circle to the diameter have the same proportion to one another, as two lines drawn from the point of intersection of the diameter and perpendiculars parallel to the ordinates of that diameter; and if the extremity of one of these lines be in the periphery of the ellipse, the other line will likewise be in the periphery of the ellipse.

LET ABC be any ellipse, AC any diameter, and D its center, and let the circle AEC be described having AC for a
dia-

diameter, and from any two points in the periphery of the ellipse, as B and F, let the semi-ordinates BG, FH, to the diameter AC, be drawn, cutting it in G and H, and from the points G and H let the lines GE and HK be drawn perpendicular to the diameter AC, cutting the periphery of the circle in E and K: I say, the semi-ordinate BG is to the semi-ordinate FH, as the perpendicular EG is to the perpendicular KH; for produce EG, and let it cut the periphery of the circle in L; then because DG is drawn through the center, cutting EL at right angles, EL is * bisected in G, * *E. I. 3.* and the rectangle contained between AG and GC is † equal to the rectangle contained between EG and GL, that is, the square of GE; for the same reason the rectangle contained between AH and HC is equal to the square of HK; wherefore as the rectangle contained between AG and GC is to the rectangle contained between AH and HC, so is the square of GE to the square of HK. But the square of BG is ‡ to the square of FH, as the rectangle contained between AG and GC is to the rectangle contained between AH and HC; therefore the square of BG is to the square

† *cor. 1. prop. 13 of this*

S 2 of

PROPOSITION XV.

THEOREM XIV.

If a circle be described having any diameter of an ellipse for its diameter, and a tangent to the circle be drawn which cuts that diameter produced, and from the point of contact a perpendicular be drawn to the diameter, and from that point of intersection an ordinate be drawn to the same diameter; and if from the point of intersection of the tangent and diameter a right line be drawn to that point of the periphery of an ellipse where the ordinate cuts it, that right line is a tangent to the ellipse.

LET ABC be any ellipse, AC any diameter, and E the center; upon the diameter AC let the circle ADC be described, and from D let DF be drawn a tangent to the circle, which meets the diameter AC produced in F, and from D let DG be drawn perpendicular to AC, and GB a semi-ordinate to AC, meeting the periphery of the ellipse in B, and join BF: I say, the line BF is a tangent to the ellipse;

lipse; for if not, let it, if possible, touch the periphery of the ellipse in another point, as H, and from H draw HK a semi-ordinate to the diameter AC, cutting it in K, and from K draw KL * perpendicular to AC, cutting the line FD in L. ^{* 11 E. I. 1.}

Then because LK is parallel to DG, the triangles FKL, FGD, are equiangular; therefore as FK is to FG, † so is KL to † 4 E. I. 6. DG; and for the same reason the triangles FKH, FGB, are similar, and as FK is to FG, so is KH to BG; consequently as BG is to HK, † so is DG to LK. But BG † 11 E. I. 6. and HK are semi-ordinates to the diameter AC, and DG and LK are perpendiculars to the same diameter AC, and D is a point in the periphery of the circle described upon the diameter AC; therefore L will likewise be a * point in the periphery of ^{* 14 of this} the same circle. But L is a point in the line FD, which is a tangent to the circle; consequently the tangent FD touches the periphery of the circle in more than one point, † which is absurd. And after the same manner may it be demonstrated, that ^{† cor. 16. E. l. 3.} if FB be a tangent to the ellipse, FD is a tangent to the circle.

Therefore, *If a circle be described having*

ving any diameter of an ellipse for its diameter, and a tangent to the circle be drawn which cuts that diameter produced, and from the point of contact a perpendicular be drawn to the diameter, and from that point of intersection an ordinate be drawn to the same diameter, and if from the point of intersection of the diameter and tangent a right line be drawn to that point of the periphery of an ellipse where the ordinate cuts it, that right line is a tangent to the ellipse; which was to be demonstrated.

PROPOSITION XVI.

THEOREM XV.

If from any point in the periphery of an ellipse a tangent be drawn meeting any diameter, and from the point of contact an ordinate be drawn to the same diameter, the rectangle contained between the segments of the diameter, intercepted between the ordinate and tangent, and the ordinate and center, is equal to a rectangle contained between the segments of the diameter intercepted between the or-
di-

ordinate and the vertexes of the diameter. Also the rectangle contained between the segments of the diameter, intercepted between its vertexes and the tangent, is equal to a rectangle contained between the segments of the diameter, intercepted between the tangent and center, and the tangent and ordinate.

LET ABC be any ellipse, E its center, AC a diameter, and from any point in its periphery, as B, let BF be drawn a tangent meeting the diameter AC produced in F, and from B let BG be drawn an ordinate to the diameter AC: I say, the rectangle contained between AG and GC is equal to the rectangle contained between FG and GE. For upon AC, as a diameter, describe the circle ADC, and from G draw GD * perpendicular to AC, and cut- * 11 E. I. 12. ting the periphery of the circle in D, and join DF and DE;

Then because FB is a tangent to the ellipse, FD is a † tangent to the circle, and † 15 of this. the angle FDE is a ‡ right angle; and be- ‡ 18 E. I. 3. cause DG is at right angles to AC, the square of DG is * equal to the rectangle * 3 & 35

T

con-

E. I. 3.

contained between AG and GC. But DG
^{* cor. 3 E.} is a * mean proportional between EG and
^{l. 6.} GF, therefore the square of DG is † equal
^{† 17 E. l. 6.} to the rectangle contained between EG
 and GF; consequently the rectangle con-
 tained between EG and GF, that is, the
 segments of the diameter intercepted be-
 tween the ordinate and center, and ordi-
 nate and tangent, are equal to the rect-
 angle contained between AG and GC, *viz.*
 the segments of the diameter intercepted
 between the ordinate and vertexes of the
 diameter; W. W. D.

Also I say, the rectangle contained be-
 tween CF and FA is equal to the rectangle
 contained between EF and FG: For (the
 construction remaining the same) because
 the triangle FDE is right angled at D, and
 DG is perpendicular to the base FE; there-
^{* cor. 3 E.} fore DE, † that is EA, is a mean propor-
^{l. 6.} tional between FE and EG; and conse-
 quently the rectangle contained between
^{† 17 E. l. 6.} FE and EG is * equal to the square of AE.
 Again, because the line FE is any how
^{† 2 E. l. 2.} cut in G, the square of FE is † equal to
 the rectangle contained between FE and
 FG, together with the rectangle contain-
 ed between FE and EG; that is, the square
 of

of FE is equal to the rectangle contained between FE and FG, together with the square of AE. But because the line AC is bisected in E, and ^xAF is added to it, the square of FE is equal to the rectangle contained between FC and FA, together with the square of AE; consequently the rectangle contained between FE and FG, together with the square of AE, is equal to the rectangle contained between FC and FA, together with the square of AE; take from both the common square of AE, and there will remain the rectangle contained between FC and FA, *vis.* the segment of the diameter, intercepted between the vertices of the diameter and the tangent, equal to the rectangle contained between FE and FG, *vis.* the segments of the diameter, intercepted between the tangent and center, and the tangent and ordinate.

Therefore, *If from any point in the periphery of an ellipse, &c.* which was to be demonstrated.

Cor. 1. From hence it follows, that the semi-diameter is a mean proportional between the segments of the diameter, intercepted between the center and tangent, and the center and ordinate; for

66. p. 2.

^xcr

F. cor. 22. l. 6.

ED

as FE is to **ED**, that is EA, so is EA to EG; and if EF be to EA as EA is to EG, **DF** is a tangent; for if not, from **B** draw a tangent cutting the diameter in M: Then because as EM is to EA, so is EA to EG; consequently as EF is to EA, so is EM to EA; therefore EM is equal to EF, a part to the whole, which is absurd.

Cor. 2. The segments of the diameter intercepted between its vertexes and the tangent, are to one another as the segments of the same diameter intercepted between its vertexes and the ordinate: For since FE is to EA as EA is to EG, by conversion it will be, as FE is to FA so is EA to AG; and by doubling the antecedents it will be, as FC, together with FA, is to FA, so is AC to AG; and by division, as FC is to FA, so is CG to GA.

Cor. 3. If CF be to FA as CG is to GA, or if CF be to EF as GF is to AF, in both these cases the line **DF** is a tangent: For first, because CF is to FA as CG is to GA, (by * composition) as CF, together with FA, is to FA, so is CA to GA; and by taking the half of the

* 17 E. 1. 5.

the antecedents it will be as EF is to FA, so is EA to GA; and by * conversion, as EF is to EA so is EA to GE; therefore the line FB is a † tangent to the ellipse. 2. Because CF is to EF as GF is to AF, and since the whole is to the whole as a part is to a part, * the residue CG is to the residue EA, as the whole CF is to the whole EF; and by conversion, as CF is to EC, or EA, so is EA to EG; therefore the line DF is a † tangent to the ellipse.

* cor. 19 E. l. 5.
† cor. 1 of this pr.
* 19 E. l. 5.
† cor. 1 of this pr.

PROPOSITION XVII.

THEOREM XVI.

If from the vertexes of two conjugate diameters of an ellipse, ordinates be drawn to any third diameter, the sum of the squares described on the segments of the diameter, intercepted between the center and each ordinate, are equal to the square described upon half the third diameter: Also the square of the segment of the diameter, intercepted between one of the ordinates and the center, will be equal

qual to the rectangle contained between the segments of the diameter intercepted between its vertexes and the other ordinate.

LET AB, CD, be two conjugate diameters of the ellipse ACB, whose center is G, and EF any third diameter, and from B and C, the vertexes of the two conjugate diameters, let BH and CK be drawn ordinates to the diameter EF: I say, the square of GF is equal to the sum of the two squares described upon GH and GK. For about the center G, with the distance GE, describe the circle ELM, and from the points H and K draw the lines HM and KL * perpendicular to the diameter EF, cutting the periphery of the circle in L and M, and from the point B let BN be drawn a tangent to the ellipse, cutting the diameter EF produced in N, and join the lines LG, MG and NM.

Then because BN is a tangent to the ellipse, MN is a † tangent to the circle; and because CD is the conjugate diameter to AB, BN is ‡ parallel to CD, and BH is parallel to CK; therefore the triangles GCK and BHN are equiangular; and as

CK

* 11 E. l. 1.

† 15 of this.

‡ cor. 1. pr.
15 of this.

Book II. *Elements of Conic Sections.* 151

CK is to HB, * so is GK to HN. But as * 4 E. 1. 6.
 CK is to HB, so is KL to HM; where-
 fore as GK is to HN, † so is KL to HM; † 14 of this
 and the angles LKG, MHN, are right-
 angles; consequently the triangles LKG,
 MHN, are † similar, and the angle HNM † 6 E. 1. 6.
 equal to the angle KGL; therefore the line
 GL is * parallel to MN. Again, because * 27 E. 1. 1.
 MN is a tangent to the circle, the angle
 GMN is a right angle, and MH is drawn
 perpendicular to the base; therefore the
 triangle GHM is † similar to the triangle † 8 E. 1. 6.
 MHN; consequently the triangle GHM
 is similar to the triangle GLK; and the
 sides GL, GM, subtending the right angles;
 are equal; therefore the side LK is † equal † 16 E. 1. 1:
 to GH, and GK to HM, and the square
 of GL is * equal to the sum of the two * 47 E. 1. 1.
 squares described upon GK and KL, that
 is, upon GK and ~~KL~~. But the square of ^{GH}
 GL is equal to the square of GF; there-
 fore the sum of the two squares described
 upon GK and GH, the segments of the
 diameter intercepted between the ordinates
 and the center, is equal to the square de-
 scribed upon half the diameter to which
 the ordinates are drawn; W. W. D.

Also I say, the square of GH is equal
 to

to the rectangle contained between EK and KF: For because the two squares described upon GK and GH are proved equal to the square of GF, and the square of GF is * equal to the square of GK, together with the rectangle contained between EK and KF; therefore the two squares described upon GK and GH are equal to the square of GK, together with the rectangle contained between EK and KF; take from both the common square of GK, and there will remain the square of GH, equal to the rectangle contained between EK and KF. After the same manner may it be demonstrated, that the square of GK is equal to the rectangle contained between EH and HF.

Therefore, *If from the vertexes of two conjugate diameters of an ellipse, ordinates be drawn to any third diameter, the sum of the squares described on the segments of the diameter, intercepted between the center and each ordinate, are equal to the square described upon half the third diameter: Also the square of the segment of the diameter, intercepted between one of the ordinates and the center, will be equal to the*

the rectangle contained between the segments of the same diameter, intercepted between the vertexes and the other ordinate; which was to be demonstrated.

Cor. 1. From hence it follows, that EF, the diameter to which the ordinates are drawn, is to OP its conjugate, as the distance of one of these ordinates from the center is to the other ordinate; because the square of EF is * to the square ^{*13 of this} of OP, as the rectangle contained between EK and KF, that is, the † square ^{† about p. 1} of GH, is to the square of KC; and therefore as the diameter EF, is ‡ to its ‡ ^{22 E. 1. 64} conjugate OP, so is GH, the distance of one ordinate from the center, to KC the other ordinate.

Cor. 2. The sum of the squares of any two conjugate diameters is equal to the sum of the squares of the two axes. Let AB and CD be the two axes, EF, GH, any other conjugate diameters, draw the ordinates HK, HN, EL and EM, and let O be the center of the ellipse; then because the two squares described upon OL and OK, are * equal to the square ^{* about p. 1} of OB, and the two squares described upon OM and ON, that is, † LE, and ^{† 24 E. 1. 71} KH,

U

KH,

KH are equal to the square of OC; therefore the four squares described upon the lines OK, KH, OL, LE, are equal to the squares described upon OB and OC. But the four squares described upon OK, KH, OL, LE, are * equal to the squares described upon OE and OH; consequently the two squares described upon OE and OH are equal to the two squares described upon OB and OC; and by quadrupling them, we will have the squares described upon AB and CD, the two axes, equal to the two squares EF and GH, the two conjugate diameters.

*47E. l. 1.

PROPOSITION XVIII.

THEOREM XVII.

If from the vertexes of two conjugate diameters of an ellipse four tangents be drawn, the figure formed by them is a parallelogram; which is equal to a parallelogram made by the tangents drawn through the vertexes of any other two conjugate diameters.

LET

LET AB and CD be two conjugate diameters of any ellipse, whose center is O, and from their vertexes A, B, C and D, the tangents PQ, QR, RS, SP, are drawn; because PS, QR, are tangents drawn from the vertexes of the same diameter, therefore they are * parallel to one * of this. another; and for the same reason PQ, RS, are parallel to one another, consequently the figure PQRS is a parallelogram; and for the same reason the figure made by the tangents drawn through the vertexes of any other two conjugate diameters is a parallelogram. Let EF and GH be two other conjugate diameters, and TV, VW, WX, XT, tangents drawn from their vertexes; I say, the parallelogram PQRS is equal to the parallelogram TVWX.

For from the vertexes H and E of the two conjugate diameters EF and HG, draw the ordinates HK, HN, EM and EL, to the two conjugate diameters AB and CD, and let the tangents VT, XT, meet the diameter AB produced in Y and Z, and complete the parallelogram NOYI, and join HY;

Then because EL is an ordinate to the diameter AB, and EY is a tangent to the ellipse,

ellipse, meeting the diameter in Y, as OY is to OB, * so is OB to OL; and because EL, HK, are ordinates to the diameter AB, drawn from the vertexes of two conjugate diameters, as OB is to OC, † so is OL to HK; then because the three magnitudes, OY, OB and OC, and the three magnitudes, OB, OL and HK, are two, and two in ordinate proportion, they will be in proportion by ‡ equality, that is, as OY is to OC, so is OB to HK, or YI, which is equal to it. But because CO is parallel to YI, the angles COB, OYI, are * equal; therefore the parallelogram COBP † is ‡ equal to the parallelogram ONIY. ‡ But the parallelogram ONIY is ‡ double the triangle OHY; wherefore the parallelogram EOHT is equal to the parallelogram ONIY; that is, the parallelogram EOHT is equal to the parallelogram COBP; and consequently the whole parallelogram PQRS is equal to the parallelogram VWXT.

Therefore, *If from the vertexes of two conjugate diameters of an ellipse, four tangents be drawn, the figure formed by them is a parallelogram, which is equal to a parallelogram made by the tan-*

tangents drawn from the vertexes of any other two conjugate diameters ; which was to be demonstrated.

PROPOSITION XIX.

THEOREM XVIII.

If from any point in the periphery of an ellipse a tangent be drawn, which produced cuts two conjugate diameters, the rectangle contained between the segments of the tangent, intercepted between the point of contact and these diameters, is equal to the square of a semi-diameter conjugate to that, which is drawn through the point of contact.

LET E be any point in the periphery of the ellipse AEB, through which the tangent ET is drawn, cutting the two conjugate diameters AB, CD, produced in the points Y and d, and let GH be conjugate to the diameter EF, drawn through the point of contact ; I say, the rectangle contained between YE and Ed is equal to the square of OG or OH ; For from E and H draw EL and HK * parallel to CD ; * 31 E. I. 1.
Then

Then because EL is parallel to dO, as
^{*2 E. l. 6.} YL is to LO, ^{*} so is YE to Ed; conse-
quently a rectangle contained between YL
and LO is similar to a rectangle contain-
ed between YE and Ed; and because EL
is parallel to KH, and EY to OH, the
triangles ELY and OKH are equiangular;
^{†4 E. l. 6.} and as LY is to YE, [†] so is OK to OH;
and because the rectangle contained be-
tween YE and Ed is similar to the rect-
angle contained between YL and LO,
whose homologous sides are YE and YL;
and the square of OK is similar to the square
of OH; therefore as the rectangle con-
tained between YL and LO is to the rect-
^{†22 E. l. 6.} angle contained between YE and Ed, [†] so
is the square of OK to the square of OH.
But the rectangle contained between YL
^{*16 of this.} and LO is ^{*} equal to the rectangle con-
tained between AL and LB, and the rect-
angle contained between AL and LB is
^{†17 of this.} [†] equal to the square of OK; wherefore
the rectangle contained between YL and
LO is equal to the square of OK; conse-
quently the rectangle contained between
^{†14 E. l. 5.} YE and Ed is [†] equal to the square of
OH.

Therefore, *If from any point in the*
peri-

periphery of an ellipse a tangent be drawn, which produced cuts two conjugate diameters, the rectangle contained between the segments of the tangent, intercepted between the point of contact and these diameters, is equal to the square of a semi-diameter conjugate to that, which is drawn through the point of contact; which was to be demonstrated.

Cor. From hence it follows, that if from any point in the periphery, a tangent be drawn cutting two diameters, and if the rectangle contained between the segments of the tangent, intercepted between these diameters and the point of contact, be equal to a square described upon the semi-diameter, which is conjugate to that, drawn through the point of contact; these diameters cut by the tangent are conjugate diameters.

PROPOSITION XX.

THEOREM XIX.

If from any point in the periphery of an ellipse an ordinate be drawn to any diameter, and from the vertex of that diameter a right line be drawn
per-

pendicular to it, and equal to its latus rectum, the square of the semi-ordinate, is equal to a rectangle contained between the absciss and the latus rectum of that ordinate, wanting in figure a rectangle similar to that rectangle which is contained between that diameter and its latus rectum.

LET AB be any diameter of the ellipse ACB, whose conjugate diameter is EF, and from the point C let CD be drawn an ordinate to it, cutting it in G, and from the vertex B let BH be drawn perpendicular to the diameter, and equal to its latus rectum, and complete the rectangle ABHK; I say, the square of CG is equal to the rectangle contained between GB and BH, wanting in figure a parallelogram similar to the rectangle ABHK: For join FIG. I. I. AH, and from G draw GL * perpendicular to AB, cutting AH in L, and KH FIG. I. I. in M, and through L draw LN † parallel to AB;

Then because BH is the latus rectum of the diameter AB, as AB is to BH, † so is † cor. 2. pr. 13 of this. the rectangle contained between AG and GB, to the square of GC. But as AB is
to

to BH, * so is AG to GL; therefore as * 4 E. 1. 6. the rectangle contained between AG and GB is to the square of GC, so is AG to GL. But as AG is to GL, † so is the † 1 E. 1. 6. rectangle contained between AG and GB, to the rectangle contained between LG and GB, because they have both the same height, viz. GB; therefore as the rectangle contained between AG and GB is to the square of GC, ‡ so is the rectangle ‡ 11 E. 1. 5. contained between AG and GB to the rectangle contained between LG and GB; consequently the square of the semi-ordinate GC is * equal to the rectangle BGLN. But * 9 E. 1. 5. the rectangle BGLN is a rectangle contained between the absciss GB, and the latus rectum BH, wanting in figure the rectangle HMLN, which is † similar to the rectangle † 24 E. 1. 6. ABHK, contained between the diameter AB and its latus rectum BH. After the same manner may it be demonstrated, that if from the vertex, BO be drawn equal to the latus rectum, tho' not at right angles to the diameter, and AO joined, and through G, GP be drawn parallel to BO, and cutting AO in P, the rectangle contained between GP and GB will be equal to the square of the semi-ordinate GC.

X

There-

Therefore, *If from any point in the periphery of an ellipse, an ordinate be drawn to any diameter, and from the vertex of that diameter, a right line be drawn perpendicular to it, and equal to its latus rectum; the square of the semi-ordinate is equal to a rectangle contained between the absciss and the latus rectum of that diameter, wanting in figure a rectangle similar to that rectangle, which is contained between that diameter and its latus rectum; which was to be demonstrated (a).*

PROPOSITION XXI.

PROBLEM II.

Two unequal right lines being given, mutually bisecting each other, and cutting one another at right angles; to describe an ellipse, whose two axes will be equal to the two given right lines.

LET

(a) From the property of the squares of the semi-ordinates being equal to the rectangle contained between their abscisses, and the latus rectum of the diameter, wanting in figure a rectangle similar to the rectangle contained between the diameter and its latus rectum, it was, that Apollonius called this curve an ellipse.

LET AB and CD be two unequal right lines, mutually bisecting each other in the point E, and cutting one another at right angles; it is required to describe an ellipse, which has the two lines AB and CD for its two axes. From C the extremity of the smaller line cut a part off, as CF, equal to AE half the greater line AB, and about the center C, with the distance CF, describe a circle cutting the line AB in the points G and H, in which let the extremities of a line be fixed, whose length is equal to the given line AB, and by means of a small pin, if the ellipse AKL be * described, the two given right lines, AB and CD, will be the two axes of it. * def. 1 of this.

For because CF is drawn through the center, cutting the line GH at right angles, GE is † equal to EH, and G and H are † 3 E. 1. 3 the two foci of the ellipse, the point E is the † center of the ellipse; and because † def. 3 of this, AE and EB are each equal to half the length of the generating thread, the periphery of the ellipse will pass through the points A and B; and because CG is equal to AE, the periphery of the ellipse will also * pass through the point C. But CE * cor. 4. 3 of this, is equal to ED, consequently it will like-

X 2

wife

wise pass through the point D ; and because AB is a diameter drawn through the foci, ^{† def. 6 of this.} of it is the † transverse axis, and the diameter CD at right angles to it is the ‡ ^{† def. 7 of this.} smaller axis ; therefore the ellipse AKL is described, which has AB and CD, the two given right lines, for its two axes ; *which was to be done.*

PROPOSITION XXII.

PROBLEM III.

With a given right line, and a given point out of it, to describe an ellipse, which will have the given right line for its transverse axis, and pass through the given point ; but it is necessary that the point be in such a position, that a line drawn from it perpendicular to the given line, cut it in a point between its extremities.

LET AB be the given right line, and K the given point out of it, in such a position, that KM, a line drawn from it perpendicular to AB, cuts it in M, a point between its extremities ; it is required to describe an ellipse, which will have AB
for

for its transverse axis, and which will pass through the point K. Find a line, as DC, the square of which will be to the square of AB, as the square of MK is to the rectangle contained between AM and MB, and let DC be so placed, as both to bisect itself and AB in E, as also to be at right angles to AB, and describe the ellipse * ^{21 of this,} ACB, which has AB and CD for its two axes, the point K will be in the periphery of the ellipse. For because the square of AB is to the square of DC, as the rectangle contained between AM and MB, is to the square of MK, † consequently the point † ^{13 of this.} K is a point in the periphery of the ellipse. Therefore there is described an ellipse ACB, which has the given right line AB for its transverse axis, and whose periphery passes through the given point K; *which was to be done.*

PROPOSITION XXIII.

PROBLEM IV.

To find the diameters, the axes, the foci, and center of a given ellipse.

LET

LET ABCD be any ellipse, it is required to find its diameters, its axes, its foci and center. Draw any two right lines, as BD and EF, parallel to one another, and terminated both ways by the periphery of the ellipse; * bisect these lines in the points G and H; join the line GH, which produce both ways until it cut the periphery of the ellipse in A and C; the line AC is a † diameter. After the same manner may any other diameter be found; and if AC be ‡ of this. bisected in G, the point G is the ‡ center of the ellipse.

To find the axis, the center being found as above. Take any point in the periphery of the ellipse, and join a line from that point to the center G; and if about G, as a center, with the distance GA, a circle be described, it must either ly wholly without the ellipse, wholly within it, or cut it. First, let it ly wholly without it, as KAL, the diameter is the greatest in the † 7 of this. ellipse, and is the * transverse axis; and if the circle, as MBN, lies wholly within the ellipse, the diameter BD is the least, and consequently the smaller axis; but if the circle, as EOF, cut the ellipse, join the line EF, and through the center G draw

draw GH perpendicular to EF, which produce until it cut the ellipse in the points A and C; I say, AC is one of the axes.

For because GH drawn through the center, cuts the line EF at right angles, it is * bisected in H; therefore EF is an † or-
* 3 E. 1. 3.
† def. 9 of this.
 dinate to the diameter AC, and is ‡ pa-
† 12 of this.
 rallel to the tangent QP, drawn through the vertex A of the diameter AC. But EHA is a right angle, consequently QAH is a * right angle; therefore the diameter
* 29 E. 1. 1.
† cor. 3. pr. 8 of this.
 AC is one of the † axes; and if BD be drawn at right angles to it, it will be the other axis. The foci may be found after the same manner, as in prop. 21 of this; wherefore the diameters, the center, the axes, of the given ellipse ABC, are found; which was to be done.

PROPOSITION XXIV.

PROBLEM V.

Two right lines being given, mutually bisecting one another, to describe an ellipse, which has the two given lines for two conjugate diameters, and to find its axes.

LET

LET AB and CD be any two given right lines, mutually bisecting each other in the point E; it is required to describe an ellipse, and to find its axes, which will have the lines AB and CD for two conjugate diameters. Produce EA, and
 * 11 E. 1. 6. find a * third proportional to AE and EC,
 † 10 E. 1. 1. which let be AF; † bisect EF in G, and
 ‡ 31 E. 1. 1. from A draw AH ‡ parallel to CD, and
 * 11 E. 1. 1. from G draw GK * perpendicular to EF, cutting the line AH in K; about the center K, with the distance KE or KF, describe a circle which cuts the line AH in the points H and L, and join the lines EL and EH, and from the point A draw AM
 † 13 E. 1. 6. perpendicular to HE, and find a † mean proportional to EM and EH, which let be EN, and make EO equal to EN; describe the ‡ ellipse NAO, which has NO for one of its axes, passing through the point A: I say, the given right lines, AB, CD, will be two conjugate diameters of that ellipse. For because A is a point in the periphery of the ellipse, and from it AM is drawn perpendicular to the axis NO, and EM is to EN, as EN is to EH; therefore AH is a * tangent to the ellipse; and because CD is parallel to HA, CD will

* cor. 1. pr.
 16 of this.

will be in † the same position with the di- † *cor. 1. pr.*
 ameter which is conjugate to AB; and be- 12 *of this.*
 cause the angle HEL is a right angle, it
 being † in a semicircle, therefore EL is † *E. I. 3.*
 the conjugate axis to NO; and because
 LH is a tangent cutting two conjugate di-
 ameters produced in H and L, the rect-
 angle contained between HA and AL is
 * equal to the square of the semi-diameter * *19 of this.*
 which is conjugate to AE, drawn through
 the point of contact. But the rectangle
 contained between LA and AH is † equal † *35 E. I. 37*
 to the rectangle contained between EA
 and AF; and since as EA is to ED, so
 is ED to AF, † therefore the rectangle † *17 E. I. 6.*
 contained between EA and AF is equal
 to the square of ED; consequently ED is
 half the diameter which is conjugate to
 AE; wherefore the periphery of the el-
 lipse NAO will pass through the point D;
 and because DE is equal to EC, and AE
 to EB, it will also pass through the points
 B and C; therefore the lines AB and CD
 are conjugate diameters to the ellipse NAO,
 which is an ellipse described, whose axes
 NO and LE are found, and which has the
 two given lines, AB and CD, mutually

Y. bi.

bisecting each other, for two conjugate diameters; *which was to be done.*

PROPOSITION XXV.

PROBLEM VI.

With a given right line, and a given point out of it, to describe an ellipse, which will have the given line a diameter, and whose periphery will pass through the given point; but the given point must be in such a position, that a line drawn from it, parallel to an ordinate to that diameter, cuts it between its extremities.

LET AB be the given right line, and P the given point, and from P let PQ be drawn parallel to an ordinate to the diameter AB, cutting it in Q, between its extremities A and B; it is required to describe an ellipse, which has AB for a diameter, and whose periphery will pass through the point P. * Bisect the line AB in E, and through E draw EC † parallel to PQ, and take EC, ED, equal to one another, and in such a manner, that the square of AE may be to the square of EC,

as the rectangle contained between AQ and QB is to the square of PQ; and describe the ellipse ACB, which has AB and CD for two * conjugate diameters: I say, ^{*24 of this.} the ellipse ACB will pass through the point P. For because from the point P a right line is drawn parallel to the ordinate of the diameter AB, and cutting it below its vertex, and the square of AB is to the square of its conjugate CD, as the rectangle contained between the segments of the diameter, intercepted between its vertexes and the line drawn cutting it, is to the square of that line; consequently the point P is a † point in the periphery of ^{†13 of this.} the ellipse. Therefore the ellipse ACB is described, which has the given line AB for a diameter, and its periphery passes through the given point P out of it; *which was to be done.*

PROPOSITION XXVI.

THEOREM XX.

If from every point in a given line, right lines parallel to one another be drawn, cutting a given right line between its extremities; and if the

Y 2 squares

squares of the parallel right lines have the same proportion to one another, as the rectangles contained between the segments of the given right line, intercepted between its extremities and these respective parallel lines; that line is the periphery of an ellipse, which has the given right line for a diameter.

LET ABC be any line, and AC a given right line, and from the points D, F and B, &c. let the lines DE, FG, BH, &c. be all drawn parallel to one another, cutting the line AC in the points E, G and H; and if the squares of DE, FG, BH, &c. have the same proportion to one another, as the rectangles contained between AE and EC, between AG and GC, and between AH and HC, &c. the segments of the given line AC, intercepted between its extremities and the points E, G and H, where these parallel lines cut it: I say, the line ABC is the periphery of an ellipse, which has the given right line AC for a diameter. For describe an ellipse, which has AC the given right

**as of this,* line for a * diameter, and DE parallel to

an ordinate of that diameter, and whose periphery will pass through the given point D: Then because DE is parallel to an ordinate of the diameter AC, all the other lines, FG, DE, &c. are * parallel to the ^{* 30 E. I. 11} ordinates of the same diameter AC; and because the square of DE is to the square of FG, as the rectangle contained between AE and EC, is to the rectangle contained between AG and GC, the segments of the diameter, intercepted between its vertexes and these respective lines, and the point D, is in the periphery of the ellipse; therefore the other point F is also † in the periphery of the ellipse. After the same manner may it be demonstrated, that any other point in the line ABC is in the periphery of an ellipse, which has the given line AC for a diameter.

† cor. 2. pr.
13 of this.

Therefore, If from every point in a given line, right lines, parallel to one another, be drawn, cutting a given right line between its extremities; and if the squares of the parallel right lines have the same proportion to one another, as the rectangles contained between the segments of the given right line, intercepted between its extremities

ties and these respective parallel lines; that line is the periphery of an ellipse, which has the given right line for a diameter; which was to be demonstrated.

PROPOSITION XXVII.

THEOREM XXI.

If any cone be cut by a plane passing through its axis, and by another plane, neither parallel to the base of the cone, nor making a subcontrary section (a), and cutting both the sides of the triangle made by the plane passing through the axis; and if the common section of the cutting plane, and the base of the cone, coincide in a right line, which is perpendicular
to

(a) Apollonius Pergaeus demonstrates, in the 5th prop. of the first book of his Conics, that if any scalene cone be cut by a plane through its axis, and perpendicular to its base, and by another plane at right angles to the triangle made by the section through its axis, and cutting off from the vertex a triangle similar to the triangle made by the plane passing through the axis, but in a subcontrary position to it, then the figure made by the common section of the cutting plane, and the surface of the cone, is a circle, and the section is called a *subcontrary section*.

to the base of the triangle made by the plane passing through its axis; the figure made by the common section of the cutting plane, and the surface of the cone, is an ellipse, having the common section of the cutting plane, and the triangle made by the plane passing through its axis, for a diameter.

LET A be the vertex of any cone, the circle BDC its base, which is cut by a plane passing through its axis, and let ABC be the triangle made by that section, and let it be cut by another plane, which is neither parallel to the base of the cone, nor in a subcontrary position, and cutting both sides of the triangle passing through the axis, and which makes, with the surface of the cone, the figure EGF, and let KH be the common section of the cutting plane, and the base of the cone, which is perpendicular to BC the base of the triangle, passing through the axis produced: I say, the figure EGF is an ellipse, which has EF the common section of the cutting plane, and the triangle passing through the axis, for a diameter.

For

176 *Elements of Conic Sections. Book II.*

For through G and O, two points in
^{* 31 E. l. 1.} the figure EGF, draw GL and OP, * parallel to KH, cutting EF in L and P, and through the points L and P draw LM, QP, † parallel to BC, cutting both ~~sides~~ of the triangle in the points M, N, Q and R ;

Then because ML is parallel to BK, and GL to KH, the plane passing through ML
^{† 15 E. l. 11.} and LG is † parallel to the plane passing through BK and KH, that is, to the base of the cone ; consequently the plane passing through ML and LG is a † circle,
^{† prop. 27. l. 1 of this.} which has MN for a diameter. Again, because the lines ML, LG, are parallel to
^{* 10 E. l. 11.} BK, KH, the angle MLG is * equal to the angle BKH, that is, a right angle ; therefore the rectangle contained between
^{† 3 & 35 E. l. 3.} ML and LN is † equal to the square of GL. After the same manner may it be demonstrated, that the square of PO is equal to the rectangle contained between QP and ~~RP~~. But the rectangle contained between ML and LN is to the rectangle
^{† 23 E. l. 6.} contained between QP and PR, in a † ratio compounded of ML to QP, and of LN to PR : But because ML is parallel to QP, the triangles EML, EQP, are similar ;

lar; and for the same reason the triangles LNF, PRF, are similar; therefore ML is to QP as * EL is to EP, and as LN is to † 4 E. 1. 61 PR, so is LF to PF; wherefore the ratio compounded of ML to QP, and of LN to PK, is equal to the ratio compounded of EL to EP, and of LF to PF, that is, as the rectangle contained between EL and LF † is to the rectangle contained between EP and PF; therefore the rectangle contained between EL and LF, ‡ is to the rectangle contained between EP and PF, as the square of GL is to † 11 E. 1. 51 the square of OP. The same may be demonstrated of any other line drawn from any other point in the figure EGF, cutting the line EF, and parallel to KH; consequently the figure EGF is * the periphery † 26 of the ellipse, which has EF, the common section of the cutting plane, and the triangle made by the plane passing through the axis, for a diameter.

Therefore, *If any cone be cut by a plane passing through its axis; and by another plane neither parallel to the base of the cone, nor making a subcontrary section, and cutting both sides of the triangle made by the plane passing*
Z
through

178 *Elements of Conic Sections. Book II.*

through the axis; and if the common section of the cutting plane and the base of the cone coincide in a right line, which is perpendicular to the base of the triangle made by the plane passing through its axis; the figure made by the common section of the cutting plane and the surface of the cone, is an ellipse, having the common section of the cutting plane, and the triangle made by the plane passing through its axis, for a diameter; which was to be demonstrated.

The end of the second book.



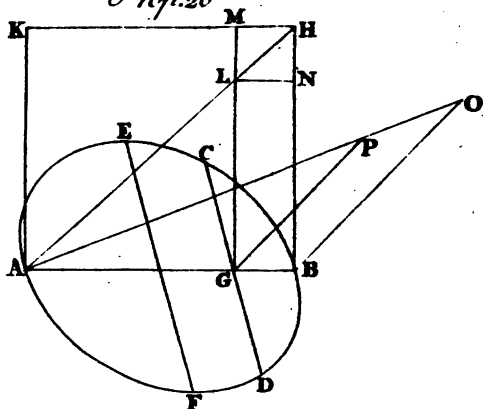
E L E

through the axis; and if the common section of the cutting plane and the base of the cone coincide in a right line, which is perpendicular to the base of the triangle made by the plane passing through its axis; the figure made by the common section of the cutting plane and the surface of the cone, is an ellipse, having the common section of the cutting plane, and the triangle made by the plane passing through its axis, for a diameter; which was to be demonstrated.

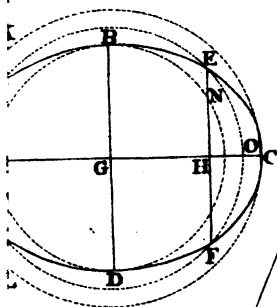
The end of the second book.



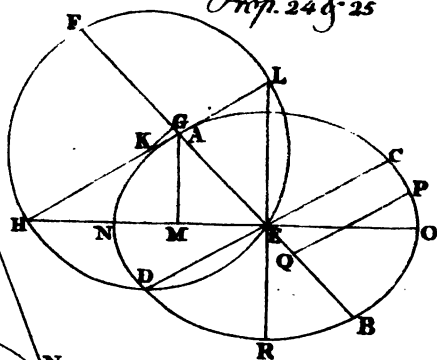
Prop. 20



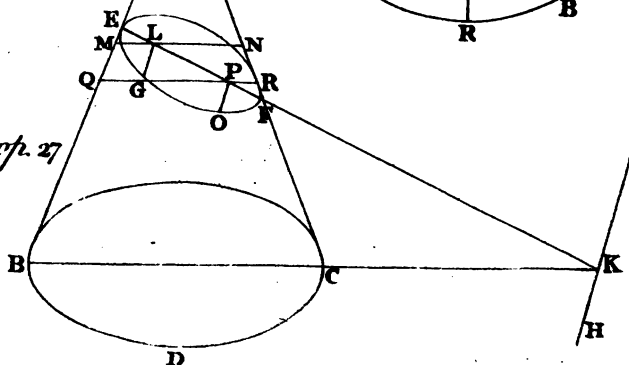
Prop. 23



Prop. 24 & 25



Prop. 27



11

ELEMENTS

OF

Conic Sections.

BOOK III.

Of the *HYPERBOLA*.

DEFINITIONS.

I. **I***F to the point A in any plane, one end of the rule AB be placed, in such a manner, that about that point, as a center, it may freely move; and if to the other end B of the rule AB, be fixed the extremity of the threed BDC, whose length is smaller than the rule AB, and the other end of the threed being fixed in the point C, coinciding with the side of the rule AB, which is in the same plane with the given point A; and let part of the threed, as BD, be brought close to the side of the rule AB, by means of a small pin D; then let the rule be moved about the point A, from C to-*
Z 2
wards

wards F, the threed all the while being extended, and the remaining part coinciding with the side of the rule, being stopt from going from it by means of the small pin, and by the motion of the small pin D, a certain figure is described which is called the semi-hyperbola.

And if the rule be brought to its first given position, and in the same manner be moved about the center A, from C towards H, the other semi-hyperbola will be described.

Also if the extremity A, of the ruler AC, be now fixed to the point C, and the end C, of the threed BDC, be fixed at the point A; and if, in the same manner as before, the rule be moved round the point C, another hyperbola will be described, which is opposite to the first, and both these together are called opposite hyperbolas.

Likewise these hyperbolas may be extended from the points C and A to a distance greater than any given distance, viz. if the length of the generating threed be taken greater than that distance; and if from any point two right lines be drawn to the points A and C, either
coin-

coinciding in a right line or not, these lines will cut the opposite hyperbolas.

II. *The two points A and C are called the foci.*

III. *The point G, which bisects the line AC, joining the foci, is called the center of the hyperbola or opposite hyperbolas.*

IV. *Any right line drawn through the center, and terminated by the opposite hyperbolas, is called a first or transverse diameter, and the points where it cuts the hyperbolas are called its vertexes; but a right line drawn through the center, bisecting a line terminated by the opposite hyperbolas, is called a second diameter.*

V. *That diameter which is drawn through the foci, is called the first, or transverse axis, and its vertexes are called the principal vertexes.*

Cor. *From this and def. 1. it follows, that if a line be drawn at right angles to the transverse axis, cutting it below its vertex, it will produced cut the periphery of the hyperbola.*

VI. *If from one of the vertexes, E, of the transverse axis EK, a part, EL,*
be

be cut off equal to GC, the distance between the center and one of the foci; and if about E, as a center, with the distance EL, a circle be described, cutting a line drawn through the center at right angles to the transverse axis in M and N, the line MN is called the second axis.

VII. Two diameters, one of which bisects all the lines that are parallel to the other, and terminated both ways by the same or opposite hyperbolas, are called conjugate diameters.

VIII. Any right line not drawn through the center, and terminated both ways by the same or opposite hyperbolas, and bisected by a diameter, is called an ordinate to that diameter; also a diameter which is parallel to an ordinate of any other diameter, is called an ordinate to that diameter.

IX. The segment of the diameter, intercepted between its vertex and an ordinate, is called the absciss of that ordinate.

X. A right line which touches the hyperbola in one point only, and produced both ways, falls without it.

is called a tangent to the hyperbola.

XI. A third proportional to two conjugate diameters is called the latus rectum, or the parameter to the diameter first mentioned in the proportion.

XII. If through the vertex of the transverse axis a right line be drawn parallel and equal to the second axis, and bisected by the transverse axis, and right lines drawn from the extremities of that line to the center, these lines are called the asymptotes of the hyperbola.

Cor. From this def. it follows, that if the extremities of the two axes be joined with right lines, these lines are parallel to the asymptotes of the hyperbola: For because ED is parallel and equal to AL, therefore AD is * parallel to LE.

* 37 E. I. 1.

XIII. Any right line drawn through, and bisected in the center of an hyperbola, parallel to a tangent, and which is equal in length to the segment of the tangent intercepted between the asymptotes, is called a second diameter to the transverse, drawn through the point of contact.

XIV. If

XIV. *If upon two right lines, AB and CD, mutually bisecting each other, and cutting one another at right angles, two opposite hyperbolas, as AF, BG, be described, and upon the same right lines two other opposite hyperbolas be described, in such a manner, that CD, the second axis of the first two, is the transverse axis of the last two; and the transverse axis of the first two is the second axis of the last two: These four hyperbolas are called conjugate hyperbolas.*

PROPOSITION I.

THEOREM I.

If from any point in the periphery of an hyperbola two right lines be drawn to the foci, the excess by which the greater exceeds the smaller, is equal to the transverse axis.

LET FDE be any hyperbola, whose foci are A and C, and its transverse axis KE, and from any point in it, as D, let the right lines, DA, DC, be drawn to the foci; I say, the excess by which

which DA the greater exceeds DC the smaller, is equal to KE the transverse axis. For let ADB be the generating rule, and BDC the generating threed, and let the small pin, with which the hyperbola was described, be at the point D; from both take DB, which is common, then will the excess by which the remainder AD exceeds the remainder DC, be equal to the excess by which the generating rule AB exceeds the generating threed BDC. The same may be proved of any other point in the periphery of the hyperbola. And because the vertexes of the transverse axis are in the periphery of the hyperbolas, therefore the excess by which KC exceeds KA, as also by which EA exceeds EC, is equal to the excess by which the rule exceeds the generating threed, that is, the excess by which AD exceeds DC; and therefore these excesses are all equal to one another; and by adding EC to both AE and EC, then will the excess by which AC exceeds twice EC, be equal to the excess by which AE exceeds EC. For the same reason the excess by which AC exceeds AK twice, is equal to the excess by which KC exceeds KA; consequently

A a

the

the excess by which AC exceeds EC twice, is equal to the excess by which AC exceeds AK twice; therefore twice AK is equal to twice EC, and AK equal to EC; wherefore the excess by which AE exceeds EC, is equal to the excess by which AE exceeds AK. But the excess by which AE exceeds AK is equal to the transverse axis EK; therefore the excess by which EA exceeds EC, that is, the excess by which AD exceeds DC, is equal to the transverse axis KE.

Therefore, If from any point in the periphery of an hyperbola two right lines be drawn to the foci, the excess by which the greater exceeds the smaller is equal to the transverse axis; which was to be demonstrated.

Cor. 1. From hence it follows, that the transverse axis is bisected in the center. For because GA is equal to GC, and AK equal to EC, therefore the remaining part GE is equal to the remaining part GK.

Cor. 2. Opposite and conjugate hyperbolas have the same right lines for their asymptotes. Let EL, EM, be the asymptotes of the hyperbola AF, and through

A and B, the vertexes of its transverse axis, draw the lines LN, ON, * paral-^{31 E. 1. 1.} lel to the second axis CD, and cutting the asymptotes in the points O, L, M and N; then because the lines BN, LA, are both parallel to CE, they are † parallel to one another, and the angle †^{30 E. 1. 1.} BNE † equal to the angle ELA; also †^{29 E. 1. 1.} the angle AEL * equal to the angle *^{15 E. 1. 1.} BEN, and the side BE equal to EA, (by the preced. cor.) therefore BN is † equal to LA, that is, CE, half the^{26 E. 1. 1.} second axis. After the same manner may OB be proved equal to CE; consequently the lines MO, LN, are the † asymptotes of the hyperbola GB; al-^{def. 12 of this.} so the lines OM and LN are the asymptotes of the other two opposite hyperbolæ CH and DK; for join LC, and produce it until it cut the asymptote MO in O, then because LA is parallel and equal to CE, CL is * pa-^{34 E. 1. 1.} rallel and equal to EA; and because the triangles OCE, EAM, are equiangular, and the side CE equal to the side AM, which subtend the equal angles AEM, COE, therefore CO is † equal to AE; ^{26 E. 1. 1.} consequently the lines OM, LN, are
A 3 2 the

* *def. 12 of this.* the * asymptotes of the hyperbola CH, and its opposite DK (a).

PROPOSITION II.

THEOREM II.

If from any point two right lines be drawn to the foci of two opposite hyperbolas, and if the excess by which the greater of these lines exceeds the smaller, be equal to the transverse axis, that point will be in the periphery of one of the opposite hyperbolas.

LET AB, EH, be two opposite hyperbolas, whose foci are D and C, BE the transverse axis, and from any point, as A, let the two right lines, AD, AC, be drawn to the foci, of which let AC be the least, and let the excess by which AD exceeds AC be equal to BE the transverse axis; I say, the point A is a point in the periphery of the hyperbola AB, whose focus C is nearest the given point: For if A be not a point in the periphery of the hyperbola AB, then will the hyperbola † cut the lines AC or AD in some other

† *def. 1 of this.*

(a) This cor. is the 15th and 17th prop. of the second book of *Apollonius Pergæus*.

ther point: and first, if possible, let it cut the line AC in any other point, as F, and join the line FD.

Then because F is a point taken in the periphery of the hyperbola, and from F the two lines FD and FC are drawn to the foci; therefore the excess by which FD exceeds FC is * equal to BE. But the two * *1 of this.* sides DF and FA are † greater than DA; † *20 E. 1. 1.* wherefore the excess by which DA exceeds AC is less than the excess by which DF and FA, taken together, exceed the same line AC; and if AF be taken from DF and FA taken together, and from AC, which is common, then will the excess, by which DF and FA taken together, exceed AC be equal to the excess by which DF exceeds FC; consequently the excess by which DA exceeds AC, is less than the excess by which DF exceeds FC. But DF exceeds FC by the transverse axis, therefore the excess by which DA exceeds AC, is less than the transverse axis, and equal to it, ‡ which ‡ *hypob.* is absurd. Neither can the hyperbola cut the line DA in any other point than A; for, if possible, let it cut it in G, and join GC; then because G is a point in the periphery of the hyperbola, the excess by which DG

ex-

exceeds GC is equal to BE the transverse axis, that is, the excess by which DA exceeds AC. But AG and GC taken together

*20E. l. 1. there are * greater than AC; consequently the excess by which DA exceeds AG and GC taken together, is less than the excess by which the same DA exceeds AC. But the excess by which DA exceeds AG and GC taken together, is equal to the excess by which DG exceeds GC; therefore the excess by which DG exceeds GC, is less than the excess by which DA exceeds AC; consequently the excess by which DA exceeds AC, is greater than the transverse axis, which is absurd; wherefore the point A is a point in the periphery of the hyperbola.

Therefore, *If from any point two right lines be drawn to the foci of two opposite hyperbolas, and if the excess by which the greater of these lines exceeds the smaller, be equal to the transverse axis, that point will be a point in the periphery of one of the opposite hyperbolas; which was to be demonstrated.*

Cor. 1. From hence it follows, that if a point be taken either between the periphery and axis, or without the periphery

phery of an hyperbola, and from it two lines be drawn to the foci, in the first case the excess by which the greater of these lines exceeds the smaller, will be greater than the transverse axis and in the second case it will be less; because the excess by which DF exceeds FC was proved * greater than *by this pr* the excess by which DA exceeds AC , and the excess by which DG exceeds GC is demonstrated to be less than the excess by which DA exceeds AC , that is; the transverse axis.

Cor. 2. If from any point two right lines be drawn to the foci of two opposite hyperbolas, and if the excess by which the greater of these lines exceeds the smaller, be either greater or less than the transverse axis; in the first case the point will either be in the axis, and below its vertex, or between the axis and periphery; and in the second case it will be without the periphery of the hyperbola.

Cor. 3. From hence it also follows, that if from the vertex of the transverse axis a line be drawn perpendicular to it, that line is a tangent to the hyperbola: For
from

from B, the vertex of the transverse axis, let BK be drawn perpendicular to it; take any point in it, as K, and join KC and KD to the foci; cut a part off the transverse axis, as BL, equal to BC, and join KL; then because BC is * equal to DE, BL is equal to DE; and therefore LD is equal to BE the transverse axis. Again, because LB is equal to BC, and BK common, and the contained angles LBK, KBC, right angles; therefore the base KL is † equal to the base KC. But because DL and LK are ‡ greater than KD, therefore the excess by which KD exceeds KL, that is, KC, is less than the excess by which DL and LK taken together, exceed the same KL; that is, LD, equal to BE the transverse axis; wherefore the excess by which KD exceeds KC, is less than the transverse axis; consequently * the point K is without the periphery of the hyperbola. And after the same manner may it be demonstrated, that every point in the line BK, except the point B, falls without the periphery of the hyperbola; therefore the line BK is a * tangent to the hyperbola.

* 1 of this.

† 4 E. I. 1.

‡ 20 E. I. 1.

* *proc. cor.*

* *def. 10 of this.*

PRO-

PROPOSITION III.

THEOREM III.

The square described upon half the second axis of any hyperbola is equal to a rectangle contained between the segments of the transverse axis, intercepted between one of the foci and its vertexes.

LET AB be any hyperbola, BF the transverse, and GH the second axis, C and D the foci; I say, the square of GE, half the second axis, is equal to the rectangle contained between FC and CB the segments of the transverse axis, intercepted between its vertexes and the focus C. For join BG;

Then because the angle BEG is a right angle, the square of GB is * equal to the sum of the squares described upon BE and EG. But GB is † equal to EC, therefore † the square of EC is equal to the sum of the two squares described upon BE and EG; and because BF is bisected in E, and BC added to it, the square of EC is ‡ equal to the rectangle contained between FC and CB, together with the square of BE;

B b

BE;

BE; therefore the rectangle contained between FC and CB, together with the square of BE, is equal to the sum of the two squares described upon BE and EG; take from both the common square of BE, and there will remain the rectangle contained between FC and CB, the segments of the transverse axis, intercepted between its vertexes and the focus C, equal to the square of GE half the second axis.

Therefore, *The square of half the second axis of any hyperbola is equal to the rectangle contained between the segments of the transverse axis, intercepted between its vertexes and one of the foci; which was to be demonstrated.*

Cor. The transverse axis is the least of all the transverse diameters in an hyperbola, and the angle contained between any other transverse diameter, and a tangent drawn from its vertex, is an acute angle: For let EB be half the transverse axis, and AE the half of any other transverse diameter; from A let AO be drawn perpendicular to the transverse axis, cutting it in O, and let AQ be a tangent to the hyperbola, touching it in A; then because EAO is a right-angled

angled triangle, AE is * greater than *¹⁹ E: L 1.
EO; consequently AE is much greater
than EB, and the angle EAO is an a-
cute angle; therefore the angle EAQ,
contained between the transverse diame-
ter and tangent, is more acute.

PROPOSITION IV.

THEOREM IV.

*If from any point in the periphery of
an hyperbola, which is not the ver-
tex of the transverse axis, a line be
drawn perpendicular to the trans-
verse axis, also a line to the nearest
focus, and from the vertex of the
transverse axis nearest to the given
point, a line be cut off the axis pro-
duced, equal to the distance between
the given point and the nearest focus;
half the transverse axis will have
the same proportion to the distance be-
tween one of the foci and the center,
as the segment of the axis, intercept-
ed between the perpendicular and the
center, is to the sum of half the trans-
verse axis, and the line drawn from
the given point to the nearest focus.*

B b 2

Also

Also the rectangle contained between the segments of the transverse axis, intercepted between the foci and that point where the line cut off it produced, equal to the line drawn from the given point to the nearest focus, cuts, exceeds the rectangle contained between the segments of the transverse axis, intercepted between its vertexes and the perpendicular, by the square of the perpendicular.

LET AB be any hyperbola, C and D the two foci, and E the center; and from any point A, in its periphery, let AO be drawn perpendicular to the transverse axis, cutting it in O, and join AC to the nearest focus C; and from B, the vertex of the transverse axis, nearest to the given point A, cut a part off it produced equal to AC, which let be BK: I say, as EB, half the transverse axis, is to EC, the distance of the focus from the center, so is EO, the distance of the center from the perpendicular, to EK, the sum of half the transverse axis, and the line drawn from the given point to the nearest focus. For join the line AD to the other focus, which
pro-

produce at pleasure, and about the center A, with the distance AC, describe the circle CBL, cutting the transverse axis again in the point L, and the line AD, in the points M and N:

Then because A is a point in the periphery of the hyperbola AB, and AC, AD, are lines drawn from that point to the foci; therefore the excess by which AD exceeds AC, is * equal to the transverse axis BF; and because AC is equal to AM, MD is the excess by which AD exceeds AC; that is, MD is equal to BF. But because AO is drawn from the center, cutting LC at right angles, LO is † equal to † 3 E. 1. 3: OC; consequently LC is double OC. But DC is ‡ double EC, therefore the remaining part DL is double the remaining part EO. Again, because MN is double AC, that is, BK, and MD equal to BF, is double BE; therefore the whole DN is double the whole EK; and because D is a point taken without the circle MBN, the rectangle contained between DC and DL is * equal to * 36 E. 1. 3: cor. the rectangle contained between ND and MD; wherefore as DM, or BF, is to DC, † so is DL to DN; and by taking their † 16 E. 1. 6: halves, it is, as EB, half the transverse axis,

xis, is to EC, the distance between the center and focus, so is EO, the distance between the center and perpendicular, to EK, the sum of half the transverse axis, and the distance from the given point to the nearest focus; W. W. D.

Also I say, the rectangle contained between DK and KC, the segments of the transverse axis, intercepted between the foci and that point where the line cuts it produced, equal to the line drawn from the given point to the nearest focus, exceeds the rectangle contained between FO and OB, the segments of the transverse axis, intercepted between its vertexes and the perpendicular, by the square of the perpendicular AO.

For because (by the above demonstration) EB is to EC as EO is to EK, the rectangle contained between EK and EB is
 * ^{16 E. l. 6.} equal to the rectangle contained between EC and EO. But because the line EK is any how cut in B, the square of EK, together with the square of EB, is
 † ^{7 E. l. 2.} equal to twice the rectangle contained between EK and EB, together with the square of BK, that is, twice the rectangle contained between EC and EO, together with the square

square of AC. But the square of AC is
 * equal to the sum of the squares described ^{47 E. I. 1.}
 upon AO and OC ; therefore the two
 squares described upon KE and EB are e-
 qual to twice the rectangle contained be-
 tween EC and EO, together with the two
 squares described upon AO and OC. A-
 gain, because the line EC is any how cut
 in O, the square of EC, together with the
 square of EO, is † equal to twice the rect- ^{† 7 E. I. 2.}
 angle contained between EC and EO, to-
 gether with the square of OC ; therefore
 the sum of the two squares described upon
 EK and EB is equal to the sum of the three
 squares described upon the lines EC, EO
 and AO ; but because DC is equally cut
 in E, and CK is added, the rectangle con-
 tained between DK and KC, together with
 the square of EC, is † equal to the square ^{† 6 E. I. 2.}
 of EK ; for the same reason the square of
 EO is equal to the rectangle contained be-
 tween FO and OB, together with the
 square of BE ; therefore the rectangle con-
 tained between DK and KC, together with
 the two squares described upon EC and
 EB, is equal to the rectangle contained
 between FO and OB, together with the
 three squares described upon EC, EB and
 AO ;

AO; take from both the sum of the two common squares described upon EC and EB, there will remain the rectangle contained between DK and KC, equal to the rectangle contained between FO and OB, together with the square of AO.

Therefore, *If any point be taken, &c.* which was to be demonstrated.

PROPOSITION V.

THEOREM V.

If from any point in the periphery of an hyperbola, which is not the vertex of the transverse axis, a right line be drawn parallel to the second axis, and cutting the transverse, the square of the transverse is to the square of the second axis, as the rectangle contained between the segments of the transverse, intercepted between its vertexes and the point where the line drawn parallel to the second axis cuts it, is to the square of the parallel line: But if the line drawn from the given point be parallel to the transverse, and cutting the second axis, the square of the second

cond is to the square of the transverse axis, as the sum of the squares of half the second axis, and that segment of it, intercepted between the center and the line drawn, is to the square of the line drawn parallel to the transverse axis.

LET AB be any hyperbola, C and D its foci, E the center, BF the transverse, and GH the second axis, and from the point A let AO be drawn parallel to the second, and cutting the transverse axis in O; I say, the square of the transverse axis BF, is to the square of the second axis GH, as the rectangle contained between FO and OB, the segments of the transverse axis, intercepted between the line drawn parallel to the second axis and its vertexes, is to the square of AO the line drawn parallel to the second axis. For join AC to the nearest focus, and make BK equal to AC;

Then because DC is equally cut in E, and CK is added, the square of EK is * 6 * 6 E. 1. equal to the rectangle contained between DK and KC, together with the square of EC; for the same reason the square of EO

C c

is

is equal to the rectangle contained between FO and OB, together with the square of
 * *4 of this.* EB. But as EB is to EC, so * is EO to
 † *cor. pr. 4.* EK; by inversion, as KE is to OE, † so
 E. l. 5. is EC to EB; consequently the square of
 ‡ *22 E. l. 6.* KE is to the square of OE, as ‡ the square
 of CE is to the square of BE; and since
 the whole, *viz.* the square of KE, is to
 the whole, *viz.* the square of OE, as a
 part taken from the first, *viz.* the square
 of CE, is to a part taken from the second,
 * *19 E. l. 5.* *viz.* the square of BE; therefore * the re-
 sidue, *viz.* the rectangle contained be-
 tween DK and KC, is to the residue, *viz.*
 the rectangle contained between FO and
 OB, as the square of KE is to the square
 of EO; consequently the square of CE is
 † *11 E. l. 5.* to the square of BE, as † the rectangle
 contained between DK and KC, is to the
 rectangle contained between FO and OB.
 ‡ *6 E. l. 2.* But the square of EC ‡ exceeds the square
 of BE, by the rectangle contained between
 FC and BC, and the rectangle contained
 * *4 of this.* between DK and KC, * exceeds the rect-
 angle contained between FO and OB, by
 the square of AO; therefore by division
 † *18 E. l. 5.* it will be, † as the rectangle contained
 ‡ *3 of this.* between FC and CB, that is, the ‡ square
 of.

of GE is to the square of BE, so is the square of AO to the rectangle contained between FO and OB; and by inversion, * as the square of EB is to the square of EG, so is the rectangle contained between FO and OB, to the square of AO, quadruple the first ratio; then will the square of BF the transverse axis, be to the square of GH the second axis, as the rectangle contained between FO and OB, the segments of the transverse, intercepted between its vertexes and the line drawn parallel to the second axis, is to the square of AO that line drawn cutting it; W. W. D.

Again, if from the given point A, AP be drawn parallel to the transverse axis FB, and cutting the second axis GH in P; I say, the square of the second axis GH will be to the square of the transverse axis FB, as the sum of the two squares of EG and EP, half the second axis, and its segment, intercepted between its center and the line drawn parallel to the transverse axis, is to the square of AP. For through the point A draw AO † parallel to the second axis, and cutting the transverse in O;

Then because AO is parallel to PE, and

C c 2

AP

- AP to OE, the figure AOEP is a parallelogram, and AO is * equal to PE. But, by the above demonstration, the square of EB is to the square of GE, as the rectangle contained between FO and OB, is to the square of AO; and, by inversion,
- 14E. l. 1. † the square of GE is to the square of BE, as the square of AO; that is, the square of PE is to the rectangle contained between FO and OB, and as one of the antecedents, *viz.* the square of GE ‡ is to one of the consequents, *viz.* the square of BE, so are all the antecedents, *viz.* the sum of the squares of GE and EP, to the sum of all the consequents, *viz.* the square of EB, together with the rectangle contained between FO and OB, that is, * the square of EO or AP; and by quadrupling the first ratio it will be, as the square of GH is to the square of FB, so is the sum of the squares of EG and EP to the square of AP.
- † cor. 4. E. l. 5.
- ‡ 12E. l. 5.
- 14E. l. 2.

Therefore, *If from any point in the periphery, &c.* which was to be demonstrated.

Cor. I. Hence it follows, that if from two or more points in the periphery of any hyperbola, lines be drawn parallel

to the second, and cutting the transverse axis, the squares of these lines are to one another, as the rectangles contained between the segments of the transverse axis, intercepted between its vertexes and these lines respectively, because they all bear the same proportion to one another, that the square of the transverse bears to the square of the second axis.

Cor. 2. If from two or more points in the same, or opposite hyperbolas, right lines be drawn parallel to the transverse, and cutting the second axis, the squares of these parallel lines are to one another, as the sum of the squares of half the second axis and its segments, intercepted between the center and these respective lines; because they are all in the same ratio of the square of the second to the square of the transverse axis.

Cor. 3. The transverse and second axes are conjugate diameters: For let AL be terminated by the same hyperbola AB in A and L, parallel to the second axis GH, and cut by the transverse in F; then because * the rectangle contain- * *prec. pr.*
ed between DF and FB is to the square
of

of AF, so is the rectangle contained between DF and FB to the square of FL; * 9 E. 1. 5. consequently the square of AF is * equal to the square of FL, and the line AF equal to the line FL. After the same manner may it be demonstrated, that any other line, terminated both ways by the same hyperbola, and parallel to the second axis, is bisected by the transverse; and if the line AC be terminated both ways by the opposite hyperbolas, parallel to the transverse, and cutting the second axis in K, then the square of EG, together with the square of EK, † is to the square of AK, as the sum of the two squares of EG and EK is to the square of KC; consequently the square of KC is ‡ equal to the square of AK, and the line KC equal to the line AK. After the same manner may it be proved, that any other line, terminated both ways by the opposite hyperbolas, parallel to the transverse, is bisected by the second axis; therefore the transverse and second axes are * *conjugate diameters*.

* def. 7 of this.

‡ *prec. cor.*

† 9 E. 1. 5.

Cor. 4. Any right line terminated both ways by the same or opposite hyperbolas, and

and bisected by one axis, is parallel to the other: For let AL be terminated both ways by the same hyperbola, and bisected in F by the transverse axis, it will be parallel to the second axis; for if not, let AN be drawn * parallel to ^{* 31 E. 1. 1.} GH the second axis, cutting the transverse in M, and the same hyperbola again in N, and through N draw NO parallel to the transverse axis, cutting the line AL in O; therefore AN is † bisect- ^{† prec. cor.} ed in M, and as AM is to MN, ‡ so ^{‡ 2 E. 1. 6.} is AF to FO; consequently AF is equal to FO. But AF is equal to FL, therefore FO is equal to FL, a part to the whole, which is absurd; therefore AL is parallel to the second axis GH. After the same manner may it be demonstrated, that a line terminated by the opposite hyperbolas, and bisected by the second, is parallel to the transverse axis.

Cor. 5. Right lines parallel to one axis, and cutting off from the other equal segments to the center, are equal to one another; and if they are equal to each other, and parallel to one axis, they cut

cut off from the other, equal segments to the center.

Cor. 6. If through the focus of an hyperbola an ordinate be drawn to the transverse axis, that ordinate is equal to the latus rectum of that axis: For let ST be drawn through the focus V, an ordinate to the transverse axis BD; then because the square of EB is to the square of EG, as the rectangle contained between DV and VB, that is, the square of GE is, * to the square of SV; wherefore * as BE is to EG, so is EG to SV; and by doubling all the terms in the proportion, as the transverse axis BD is to the second axis GH, so is the second, or conjugate axis GH, to the ordinate ST; consequently ST is the † latus rectum of the transverse axis.

* 3 of this.

* 22 E. I. 6.

† def. 11 of this.

PROPOSITION VI.

THEOREM VI.

If from any point a right line be drawn perpendicular to the transverse axis, cutting it below its vertex, and if the square of the transverse axis be to the square of the second axis, as the
the

the rectangle contained between the segments of the transverse axis, intercepted between the perpendicular and its vertexes, is to the square of the perpendicular, that point is a point in the periphery of the hyperbola. Also if from the given point a right line be drawn perpendicular to the second axis, and if the square of the second axis be to the square of the transverse axis, as the sum of the squares described upon half the second axis and its segment, intercepted between the center and perpendicular, is to the square of the perpendicular, that point is a point in the periphery of one of the opposite hyperbolas.

LET NB, DP, be two opposite hyperbolas, whose transverse axis is BD, and second axis GH, and from any point, as A, let AF be drawn perpendicular to the transverse axis BD, cutting it below the vertex in F; and if the square of DB be to the square of GH, as the rectangle contained between DF and FB is to the square of FA; I say, the point A is a point

D d in

in the periphery of the hyperbola BN. For if A is not a point in the periphery of the hyperbola BN, the hyperbola will * cut the line AF in some other point, which, if possible, let be Q.

Then because Q is a point in the periphery of the hyperbola, and from it QF is drawn parallel to the second axis, cutting the transverse in F, the square of DB † is † to the square of GH, as the rectangle contained between DF and FB is to the square of QF. But the square of DB ‡ is to the square of GH, as the rectangle contained between DF and FB is to the square of FA; therefore the rectangle contained between DF and FB * is to the square of FA, as the same rectangle contained between DF and FB is to the square of QF; † 9 E. 1. 5. consequently the square of FA † is equal to the square of FQ, and the line FA equal to the line FQ, a part to the whole, which is absurd; therefore the point A is a point in the periphery of the hyperbola BN. After the same manner may it be demonstrated, that if from the point A, AK be drawn perpendicular to the second axis, or, which is the same thing, parallel to the transverse axis, and if the square of

of

of the second axis be to the square of the transverse axis, as the sum of the squares EG and EK is to the square of AK, the point A is in the periphery of one of the opposite hyperbolas.

Therefore, *If from any point, &c.* which was to be demonstrated.

Cor. Hence it follows; that a right line drawn from any point in the transverse axis below its vertex, can cut the periphery of the hyperbola in no more than two points: For let FQ be drawn from the point F in the transverse axis below its vertex, perpendicular to it, which produce both ways, and let FA, FL, be taken on each side of the axis, equal to each other, in such a manner, that the square of the transverse axis BD, is to the square of the second axis GH, as the rectangle contained between DF and FB, the segments of the transverse axis, intercepted between its vertexes and the perpendicular, is to the square of FA or FL, the points A and L will be in the periphery of the hyperbola. Neither can the line cut it in any more points than A and L; for, if possible, let it cut it in a third point Q, then will

the rectangle contained between DF and FB, be * to the square of AF, as the same rectangle contained between DF and FB, is to the square of FQ; therefore the square of FQ is † equal to the square of FA, and the line FQ equal to the line FA, a part to the whole, which is absurd.

* cor. 1. pr
5 of this.

† 9 E. 1. 5.

PROPOSITION VII.

THEOREM VII.

If through any point in the periphery of an hyperbola, a right line be drawn parallel to the second axis, cutting both asymptotes, the rectangle contained between the segments of that line, intercepted between the asymptotes and that point, is equal to the square described upon half the second axis.

LET AB be any hyperbola, C the center, BD the transverse, and EF the second axis, CG and CH its asymptotes, and from any point in it, as A, let AH be drawn parallel to EF, cutting the asymptotes in G and H; I say, the rectangle

angle contained between GA and AH, is equal to the square of EC half the second axis. For let GH cut the axis in L, and through B, the vertex of the transverse axis, draw BM * parallel to the second axis, cutting the asymptotes in M and N; ^{*31 E. I. 1.}

Then because MN is drawn through the vertex of the transverse, parallel to the second axis, cutting the asymptotes in M and N, MB, BN, are † equal to one another, and each equal to half the second axis EC; and the two triangles CBM, CBN, are ‡ similar and * equal, and for the same reason the triangles CLG, CLH, are similar and equal, and GL equal to LH; therefore GH, the line intercepted between the asymptotes, and parallel to the second, is bisected by the transverse axis; and because the triangles CBM, CLG, are equiangular, as CB is to BM, † so is CL to LG; consequently ‡ as the square of CB is to the square of BM, so is the square of CL to the square of LG. But because A is a point in the periphery of the hyperbola, and from it AL is drawn cutting the transverse, and parallel to the second axis, as the square of CB is to the square of CE, or BM, which is equal to it, ^{† def. 12 of this.} ^{‡ 4 E. I. 1.} ^{4 E. I. 6.} ^{† 4 E. I. 6.} ^{‡ 22 E. I. 6.}

214 *Elements of Conic Sections. Book III.*

\dagger 5 of this, it, * so is the rectangle contained between DL and LB, to the square of LA; and as one of the antecedents is to one of the
 \dagger 12 E. l. 5. consequents, \dagger so are all the antecedents to all the consequents; that is, as the square of CB is to the square of BM, so is the square of CB, together with the rectangle contained between DL and LB, that is, (because the line DB is equally cut in
 \dagger 6 E. l. 3 C, and BL added to it) the \dagger square of CL to the sum of the squares of BM and LA; therefore as the square of CL is to
 \dagger 11 E. l. 4. * the square of LG, so is the same square of CL to the sum of the squares described upon BM or CE, and AL. But because GH is equally cut in L, and unequally
 \dagger 6 E. l. 2. cut in A, the square of GL is \dagger equal to the rectangle contained between GA and AH, together with the square of AL; wherefore the two squares of EC and AL are equal to the rectangle contained between GA and AH, together with the square of AL; take away the common square of AL from both, and there will remain the square of EC half the second axis, equal to the rectangle contained between GA and AH.

Therefore, *If from any point in the peri-*

periphery of an hyperbola, a right line be drawn parallel to the second axis, cutting both assymptotes, the rectangle contained between the segments of that line, intercepted between the assymptotes and that point, is equal to the square described upon half the second axis; which was to be demonstrated.

PROPOSITION VIII.

THEOREM VIII.

If any two points in the same or opposite hyperbolas be joined with a right line, which produced both ways cuts the assymptotes, the rectangle contained between the segments of that line, intercepted between the assymptotes and one of the given points, is equal to the rectangle contained between the segments of the same line, intercepted between the other given point and assymptotes; also the segments of the line intercepted between each given point and the nearest assymptote, are equal to one another.

LET

be drawn parallel to one another, cutting the asymptotes, the rectangles contained between the segments of these lines, intercepted between the given points and the asymptotes, will be equal to one another; but if one of these two lines pass through the center, that is, if it be a first, or transverse diameter, the diameter is bisected in the center, and the rectangle contained between the segments of the other line, intercepted between the given point and asymptotes, will be equal to the square of half the transverse diameter.

LET A and B be two points in the same or opposite hyperbolas, and through them the lines AC, BE, are drawn parallel to one another, and cutting the asymptotes in the points C, D, E and F; I say, the rectangle contained between AC and AD, the segments of the line intercepted between the point A and the asymptotes, is equal to the rectangle contained between BE and BF, the segments of the other line intercepted between the given point B and the asymptotes. For
through

Book III. *Elements of Conic Sections.* 211

through the points A and B draw the lines GH and LK, * parallel to the second axis, cutting the asymptotes in the points G, H, K and L;

Then because GA is parallel to BK, and AC to BE, the triangles GAC, BKE, are equiangular; and as GA is to BK, ‡ so is ‡ AC to BE. But because the lines GH and KL are parallel to the second axis, cutting the asymptotes in the points G, H, K and L, the rectangle contained between GA and AH is equal to the rectangle contained between KB and BL, because they are both ‡ equal to the square of half the second axis; therefore as GA is to KB, * so is BL to AH; consequently as AC is to BE, ‡ so is BL to AH. Again, because BL is parallel to AH, and AD to BF, the triangles ADH and BFL are ‡ equiangular, and as BL is to AH, * so is BF to AD; therefore as AC is to BE, so is BF to AD; consequently the rectangle contained between AC and AD is ‡ equal to the rectangle contained between BE and BF. Again, if any of these lines, as AC, pass through the center O, cutting the opposite hyperbola in M, the transverse diameter AM is bisected in the center; be-

E c 2

cause

cause the points C and D, cutting the asymptotes, and parallel to BF, coincide with the center, and the rectangle contained between BE and BF will be equal to the rectangle contained between AC and AD, that is the square of AD. For the same reason, if BF cut the hyperbola in N, the rectangle contained between NE and NF is equal to the square of MO. But the rectangle contained between NE and NF is equal to the rectangle contained between BE and BF, (by the above;) therefore the square of AO is equal to the square of MO, and the line AO equal to the line MO; consequently the transverse diameter AM is bisected in the center, and the rectangle contained between BE and BF equal to the square of half the transverse diameter drawn parallel to it.

Therefore, *If through any two points, &c. which was to be demonstrated.*

* fig. 1. to
this prop.

Cor. I. * If the lines CD and EF cut the same or opposite hyperbolas again in the points M and N, then will the rectangle contained between AC and CM, be equal to the rectangle contained between BE and EN, the segments of the lines intercepted between the two points where

where they cut the same or opposite hyperbolas, and one of the affymptotes; because CA is * equal to MD, therefore the rectangle contained between AC and CM, is equal to the rectangle contained between AD and AC, that is, the rectangle contained between BE and FB; and because BE is equal to ~~FN~~, the rectangle contained between BE and EN is equal to the rectangle contained between BE and BF; therefore, the rectangle contained between BE and EN is equal to the rectangle contained between AC and CM.

Cor. 2. From hence it also follows, that any transverse diameter is less than a right line drawn parallel to it, and terminated both ways by the opposite hyperbolas; because the rectangle contained between BE and BF, that is, the rectangle contained between BE and EN, is equal to the square of AO; therefore as BE is to AO, † so is AO † ^{17 E. 1} to EN; and consequently BE and EN added together, that is BN, is * great- ^{* 25 E. 1} er than AM the transverse diameter.

Cor. 3. If in the line BN, terminated both ways by the opposite hyperbolas, and paral-

parallel to the transverse diameter AM, the points E and F be taken in such a manner, that the rectangle contained between BE and EN, and between BF and FN, be equal to the square of AO half the transverse diameter, the points E and F are in the asymptotes; for if from both there be taken the common rectangle contained between BE and FN, there will remain the rectangle contained between BE and EF, equal to the rectangle contained between EF and FN; therefore BE is equal to FN; and if one of these points, ~~as B~~, be not in the asymptotes, as E, let any other point, as P, be in it, then is FN equal to BP. But BE is * equal to FN, therefore BE is equal to BP, a part to the whole, which is absurd.

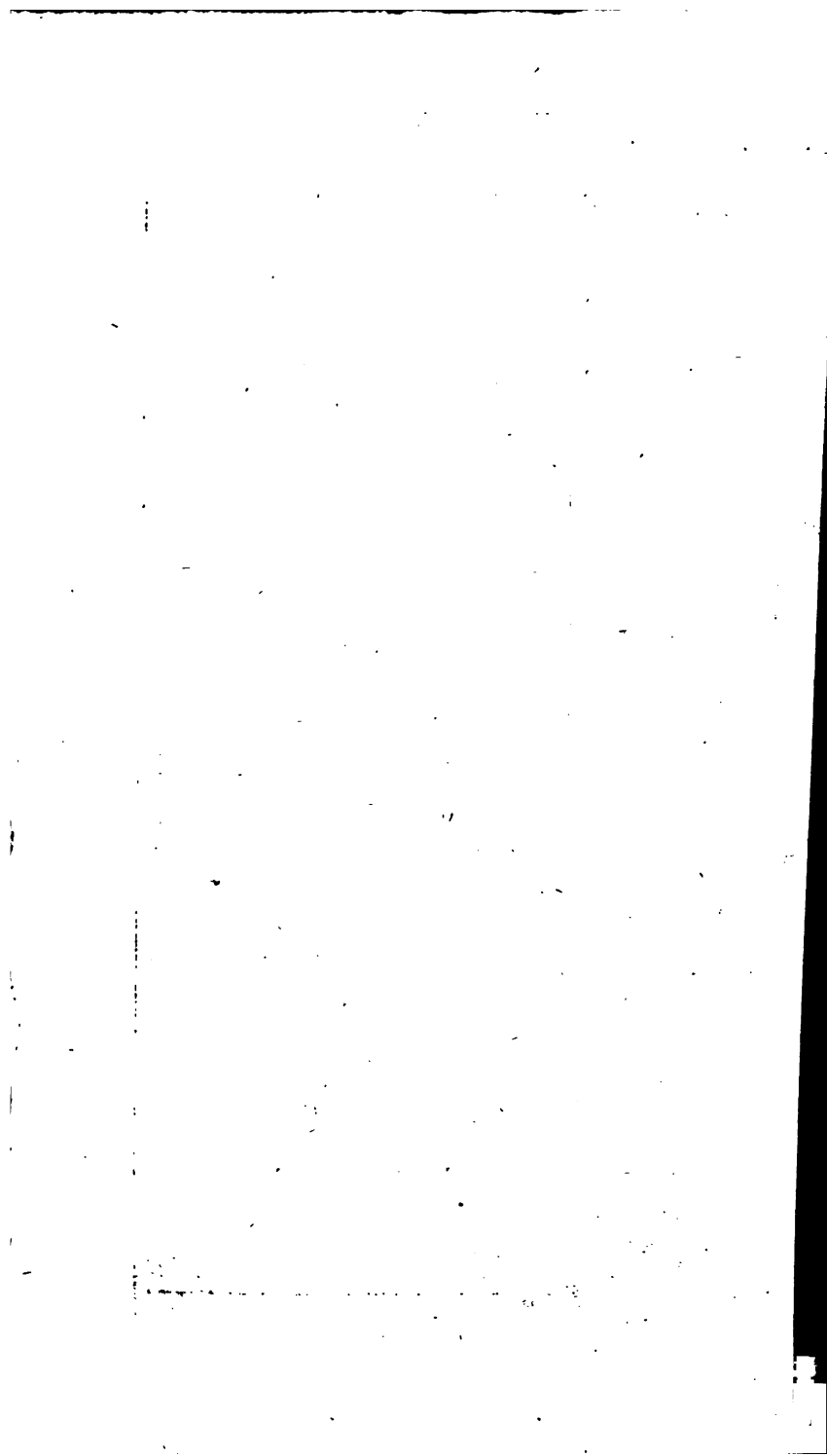
* & of this,

PROPOSITION X.

THEOREM X.

If from any point in an hyperbola two right lines be drawn, any how cutting both asymptotes, and from any other point in the same or opposite hyperbolas, two right lines be drawn
paral-

11



al to the former, and cutting
symptotes, the rectangle con-
tained between the first two lines is
to the rectangle contained be-
the last two lines. Also if
two points in the same or oppo-
hyperbolas, two right lines be
on, cutting the same or both as-
ptotes, and parallel to the other
ptote, the rectangles contained
ween these lines and the segments
the asymptotes, intercepted be-
en them and the center, are equal
ne another.

Let A and B be any two points taken in the same or opposite hyperbolas, and from the point A draw any two right lines, as AC and AD, cutting the asymptotes in C and D, and from the other point B let the lines BE and BF be drawn parallel to AC and AD, cutting the asymptotes in E and F; I say, the rectangle contained between AC and AD is equal to the rectangle contained between BE and BF. For through the points A and B draw the lines GH and KL * parallel to the second axis, cutting the asymptotes

224 *Elements of Conic Sections*

ptotes in the points G, H ,

Then because GA is parallel
 $\dagger 29 E. 1. 1.$ AC to BE , the triangles GAC
 * equiangular; and as GA is to
 AC to BE . For the same reason
 angles BFL, ADH , are similar
 BL is to AH ; so is BF to AD .
 cause the lines KL and GH are parallel
 the second axis, the rectangle contained
 $\dagger 29 E. 1. 1.$ between GA and AH is \dagger equal
 rectangle contained between KB and
 $\dagger 16 E. 1. 6.$ therefore as GA is to KB , \dagger so is
 $\dagger 11 E. 1. 5.$ AH ; consequently as AC is to BE
 is BF to AD ; wherefore the rectangle
 $\dagger 16 E. 1. 6.$ contained between AC and AD is \dagger
 to the rectangle contained between
 and BF .

Again, if A and B be two points in
 same or opposite hyperbolas, and from
 them the lines AC, BE or BF , are drawn
 cutting the same or both asymptotes, and
 parallel to the other asymptote; I say
 the rectangle contained between AC and
 the line drawn parallel to the asymptote, and
 CM the segment of the asymptote, intercepted
 between the line and center, is equal
 to the rectangle contained between
 and BF .

the points G and F . For complete the parallelogram GA is $ACMD$ and $BEMF$;

the triangles AD is equal to CM , and MF , the rectangle contained between

For the AD and AC is equal to the rectangle ADH , and contained between AC and CM , and

, so is BF rectangle contained between BE and KL and GE equal to the rectangle contained between

the BF and FM . But, by the above demonstration, the rectangles contained

between the lines AC and AD , and the rectangles BE and BF , are equal to one another;

; consequently the rectangle contained between AC and CM , is equal to the rectangle contained between BF and FM .

Therefore, *If from any point, &c.* which was to be demonstrated.

Prop. 1. Because the rectangles contained between AC and CM , and BF and FM ,

are equal to one another, as AC is to BF , * so is FM to CM ; and the paral-

lelograms $ACMD$ and $BFME$ are equiangular, therefore they are † equal to

one another.

Prop. 2. If from any point in the periphery of one of the conjugate hyperbolas,

a right line be drawn parallel to one of the asymptotes, and cutting the other,

Ff and

ptotes in the points G, H, K and L;

Then because GA is parallel to KB, and AC to BE, the triangles GAC, KBE, are
 *29 E. I. 1. * equiangular; and as GA is to KB, so is AC to BE. For the same reason the triangles BFL, ADH, are similar, and as BL is to AH, so is BF to AD. But because the lines KL and GH are parallel to the second axis, the rectangle contained
 † 7 *Whis.* between GA and AH is † equal to the rectangle contained between KB and BL;
 † 16 E. I. 6. therefore as GA is to KB, † so is BL to
 * 11 E. I. 5. AH; consequently as AC is to BE, * so is BF to AD; wherefore the rectangle
 † 16 E. I. 6. contained between AC and AD is † equal to the rectangle contained between BE and BF.

Again, if A and B be two points in the same or opposite hyperbolas, and from them the lines AC, BE or BF, are drawn, cutting the same or both asymptotes, and parallel to the other asymptote; I say, the rectangle contained between AC the line drawn parallel to the asymptote, and CM the segment of the asymptote, intercepted between the line and center, is equal to the rectangle contained between
 BF

BF and FM. For complete the parallelograms ACMD and BEMF;

Then because AD is equal to CM, and BE to MF, the rectangle contained between AD and AC is equal to the rectangle contained between AC and CM, and the rectangle contained between BE and BF is equal to the rectangle contained between BF and FM. But, by the above demonstration, the rectangles contained between the lines AC and AD, and the lines BE and BF, are equal to one another; consequently the rectangle contained between AC and CM, is equal to the rectangle contained between BF and FM.

Therefore, *If from any point, &c.* which was to be demonstrated.

Cor. 1. Because the rectangles contained between AC and CM, and BF and FM, are equal to one another, as AC is to BF, * so is FM to CM; and the parallelograms ACMD and BFME are equiangular, therefore they are † equal to † one another.

Cor. 2. If from any point in the periphery of one of the conjugate hyperbolas, a right line be drawn parallel to one of the asymptotes, and cutting the other,

F f and

and, from a point in one of the adjacent hyperbolas, a line be drawn parallel to either asymptote, and cutting the other, the rectangle contained between these lines and the segments of the asymptotes, intercepted between them and the center, are equal to one another. For let AB, DE, FG and HK, be conjugate hyperbolas, their center C, and their asymptotes MC and CL, and from the point B, in the hyperbola AB, let BM be drawn parallel to the asymptote CL, cutting the other in M, and from G, a point in the periphery of the adjacent hyperbola, let GN be drawn parallel to the asymptote CL, and cutting the other in N, the rectangle contained between BM and MC is equal to the rectangle contained between GN and NC. For let AD, FH, be the two axes of the conjugate hyperbolas, and join FA, which is * parallel to the asymptote CL, and which cuts the asymptote CM in Q, and through A draw AO † parallel to FH, cutting the asymptotes in O and L; then because AO is ‡ parallel and equal to FC, therefore the triangles FQC,

* *cor. def.*
12 of *this.*

† 31 E. I. 1.

‡ *def.* 12 of
this.

FQC, AQO, are similar and equal, and FQ equal to QA; wherefore the rectangle contained between FQ and QC, is equal to the rectangle contained between AQ and QC; but the rectangle contained between AQ and QC, is * equal to the rectangle contained be- * *prec. pra* tween BM and MC. For the same reason the rectangle contained between FQ and QC, is equal to the rectangle contained between GN and NC; consequently the rectangle contained between GN and NC, is equal to the rectangle contained between BM and MC.

PROPOSITION XI.

THEOREM XI.

Any right line drawn through the center of an hyperbola, within the angle contained between its asymptotes produced, will meet the hyperbola.

LET AB be any hyperbola, C its center, CD and CE the asymptotes, and GA the transverse axis, and through the center C let FC be drawn within the angle DCE, contained between the asymptotes;

F f 2

I say,

228 *Elements of Conic Sections. Book III.*

I say, the line CF produced will meet the
 * 31 E. l. 1. hyperbola. For through A draw AD * parallel to the second axis, cutting the asymptotes in D and E, and the line CF in L, and draw AF parallel to CD;

Then because the two angles FCG, FCA, are equal to two right angles, wherefore the two angles DCG, FCA, are less than two right angles; but the angle FAC is
 † 29 E. l. 1. † equal to DCG, therefore the two angles FAC, FCA, are less than two right angles; consequently the two lines CF and FA produced will meet, which let be in the point F, the point F will be between the periphery of the hyperbola and its axis; for, if not, it will either be in the periphery or without it.

First, it cannot be in the periphery: For through F draw HK parallel to DE, cutting the asymptotes in H and K; then because DA is parallel to HF, and HD to AF, the figure ADHF is a parallelogram, and DA is equal to HF. But the rectangle
 † 7 of this, contained between HF and FK, is † equal to the rectangle contained between DA and AE; wherefore FK is equal to AE. But FK is greater than LE, because of the similar triangles CLE, CFK; wherefore
 AE

AE is greater than LE, a part greater than the whole, which is absurd; therefore the point F is not in the periphery of the hyperbola.

Neither can F be without the periphery; for, if it be, the line HF, parallel to DA, that is, perpendicular to the transverse axis below its vertex, will cut the * periphery, which let be in M; then is * *cor. def. of this.* HK greater than HF, that is DA, and MK greater than AE; therefore the rectangle contained between HM and MK, is greater than the rectangle contained between DA and AE; and they are also † equal to one another, which is absurd; † *of this;* therefore the point F is between the axis and the periphery. But C is without the periphery, consequently the line CF produced cuts the periphery of the hyperbola.

Therefore, *Any right line drawn through the center of an hyperbola, within the angle contained between its asymptotes produced, will meet the hyperbola; which was to be demonstrated.*

PRO-

PROPOSITION XII,

THEOREM XII,

If from any point in one of the asymptotes of an hyperbola, which is not the center, a line be drawn parallel to the other asymptote, that line produced will meet the hyperbola in one point only (a).

LET CN, CO, be the asymptotes of any hyperbola, and P any point taken in one of them, which is not the center, and from P let PQ be drawn parallel to the other asymptote CO; I say, PQ produced will cut the hyperbola in one point only. For take any point, as R, in the periphery of the hyperbola, and from R draw the lines RN and RS, each
 *^{31 E. I. 1.} parallel to one of the asymptotes CO and CN, and cutting the other in N and
 †^{12 E. I. 6.} S; find a † fourth proportional to the three lines CP, NR and RS, which let be PQ, join the line CQ, the point Q will be in the periphery of the hyperbola;
 for

(a) This prop. is the 13th of the second book of Apollonius Pergaeus's Conics.

for, if not, it must either be without it or within it; and first, if possible, let it be without it, then the line CQ will * cut ^{*11 of this.} the periphery, which let be in T, and through T draw TV parallel to the asymptote CO, and cutting the other, CN, in V.

Then because TV and RS are drawn from the points T and R in the periphery of the same hyperbola, parallel to the asymptotes, the rectangle contained between TV and VC, is † equal to the ^{†10 of this.} rectangle contained between RS and SC, or NR, which is equal to SC. But because PC is to NR, as RS is to PQ, the rectangle contained between PC and PQ, is ‡ equal to the rectangle contained between ^{‡16 E.I. 6;} NR and RS; consequently the rectangle contained between PC and PQ, is equal to the rectangle contained between CV and VT. But VC is greater than PC, and VT is greater than PQ; therefore the rectangle contained between CV and VT, is greater than the rectangle contained between PC and PQ; but before it was proved equal to it, which is absurd; therefore the point Q is not without the periphery of the hyperbola. And after the same manner may it be demonstrated, that
the

the point Q is not within the hyperbola; consequently the line PQ meets the hyperbola in Q ; neither can it meet it in any other point; for, if possible, let it be in X , and through X draw XO parallel to the asymptote NC , and cutting the other asymptote CO in O ; then will the rectangle contained between XO and OC ,
 *10 of this. be * equal to the rectangle contained between QP and PC , a part to the whole, which is absurd; consequently the line QP cannot cut the periphery of the hyperbola in any other point than Q .

Therefore, *If from any point in one of the asymptotes of an hyperbola, which is not the center, a line be drawn parallel to the other asymptote, that line produced will meet the hyperbola in one point only; which was to be demonstrated.*

Cor. 1. From the above demonstration it follows, that if from any point in one of the asymptotes, a right line be drawn parallel to the other, and any point taken in that line; and if the rectangle contained between the segments of the line, intercepted between the given point and asymptote, and the segment of the
 asymptote

assymptote, intercepted between it and the center, be either greater, equal to, or less than the rectangle contained between a line drawn from any point in the periphery of the hyperbola, parallel to one assymptote, and cutting the other, and the segment of the other assymptote, intercepted between the cutting line and center; in the first case the point is within the periphery, in the second it is in it, and in the third case it is without the periphery.

Cor. 2. From hence it also follows, that any right line drawn from the center, within the angle contained between the assymptotes, cuts the periphery of the hyperbola in one point only; for if CQ cuts it in any other point, as T, the rectangle contained between TV and VC would be equal to the rectangle contained between QP and PC, which is proved to be absurd.

Cor. 3. If any right line cut the same or opposite hyperbolas in two points, it will cut both the assymptotes; for if it be parallel to any one of them, it will cut the same or opposite hyperbolas in one point only.

G g

Cor.

Cor. 4. If from any point in the periphery of one of the conjugate hyperbolas, a line be drawn parallel to one of the asymptotes, and cutting the other; and if a point be taken within the angle contained between the asymptotes of one of the hyperbolas, adjacent to the first mentioned one, and from it a line be drawn parallel to one asymptote, and cutting the other; and if the rectangle contained between these lines and their respective segments of the asymptotes, intercepted between the center and them, be equal to one another, that point is a point in the adjacent hyperbola. Let B be a point in the hyperbola AB , and G a point assumed within the angle MCP , contained between the asymptotes of the hyperbola FR , adjacent to AB , and let the rectangle contained between BM and MC , be equal to the rectangle contained between GN and NC , G will be a point in the periphery of the hyperbola FR : For let the construction be the same as in *cor. 2. prop. 10* of this; then because the rectangle contained between AQ and QC , is equal to the rectangle contained

ained between FQ and QC, and the rectangle contained between AQ and QC, is
 * equal to the rectangle contained between ^{*10 of this.} GN and NC; therefore the rectangle contained between FQ and QC, is equal to the rectangle contained between GN and NC; consequently G is a † point in the periphery of one of the hyperbolas adjacent to AB. ^{† cor. 1 of this pr.}

PROPOSITION XIII.

THEOREM XIII.

The distance between an hyperbola and its asymptotes, infinitely produced, continually decreases, and will be less than any given distance, but will never meet.

LET AB be any hyperbola, C its center, and CD, CE, its asymptotes; I say, if the hyperbola AB, with its asymptotes CD, CE, be infinitely produced, their distance from each other will continually decrease, and will be less than any given distance H, but will never meet. For through the points A and B draw the lines AE, BG, parallel to one another,
 G g 2 and

and cutting both asymptotes in the points D, E, F and G, and join AC, which produced cuts FG in K;

Then because the lines DE and FG are drawn from the points in the periphery of the same hyperbola, parallel to one another, and cutting the Asymptotes in D, E, F and G, the rectangle contained between DA and AE, is * equal to the rectangle contained between FB and BG;

*10 of this. wherefore as GB is to EA, † so is AD to BF; but because of the similar triangles KCG and ACE, KG is greater than AE, and BG is greater than KG; therefore BG is much greater than AE, wherefore DA is greater than BF. After the same manner may it be demonstrated, that the distance between the hyperbola and its asymptotes, if produced, will be less than BF; therefore the distance between the hyperbola and its asymptotes continually decreases. Again, if DL be taken smaller

‡ 51 E, l. 1. L, LM be drawn ‡ parallel to CD, that
*12 of this, line LM produced will * meet the periphery of the hyperbola, which let be in M, and through M draw MN parallel to AE, cutting the asymptotes in N and O; then

then because MO is parallel to DL, and DO to LM, the figure DLMO is a parallelogram, and MO is equal to DL; that is, MO, the distance between the periphery of the hyperbola, at the point M and the asymptote AD, is less than the given distance H.

Also I say, the asymptote and hyperbola never can meet; for, if possible, let them meet in P, and let Q be the vertex of the transverse axis, and from Q and P, the lines QR and PS are drawn parallel to the second axis, let PS cut the transverse axis in S, and let T be the other vertex of the transverse axis; then because of the similar triangles CQR, CSP, as CQ is to QR, * so is CS to SP; where- * 4 E. 1. 6.
fore as the square of CQ is to the square of QR, that is, † half the second axis, so † *def. 12 of this.* is the square of CS to the square of SP.
But as the square of QR is to the square of QC, ‡ so is the rectangle contained ‡ *of this.* between QS and ST, to the square of SP; therefore as the square of CS is to the square of SP, * so is the rectangle con- * 11 E. 1. 5.
tained between QS and ST, to the square of the same line SP; consequently the rectangle contained between QS and ST, is
* equal

* 9 E. 1. 5. * equal to the square of CS, a part to the whole, which is absurd; wherefore the periphery of the hyperbola and its asymptotes do not meet in the point P. And after the same manner may it be demonstrated, that they cannot meet in any other point.

Therefore, *The distance between an hyperbola and its asymptotes, infinitely produced, continually decreases, and will be less than any given distance, but will never meet; which was to be demonstrated (a).*

PROPOSITION XIV.

THEOREM XIV.

If from any point in the periphery of an hyperbola, a right line be drawn cutting both the asymptotes, and from either of these points of intersection a part be cut off, equal to the segment of the line intercepted between the given point and the other asymptote, in such a manner, that the ex-
tra-

(a) This prop. is the 14th of the second book of Apollonius Pergensis's Conics, and Claudius Richardus's 1st cor. to that prop.

This Proposition is also the 12th of the III Book of Simon's Sect. Con. & the 4. Cor. to the 1st Prop. of the III Book of l'Hopital's Traité Anal. des Sections Coniques.

tremity of the line cut off and the given point, are either both within, or both without the asymptotes; in the first case that other point will be in the periphery of the same hyperbola with the first given point, and in the last case it will be in the periphery of its opposite hyperbola.

LET A be any point in the periphery of an hyperbola, E the center, and EB, EC, the asymptotes, and from A let AC be drawn cutting both the asymptotes in B and C; and if from either of these points, as C, CD be cut off equal to AB, so as the points A and D are either both within or both without the asymptotes; in the first case, I say, the point D is in the periphery of the same hyperbola with the given point A, and in the last case it is in the opposite hyperbola. For through the points A and D draw the lines AF and DG * parallel to the asymptote EC, and cutting the other asymptote EB in F and G;

Then because DG is parallel to EC, as DC is to GE, † so is BD to BG, and as †² E. l. 6. BD is to BG, † so is BA to BF; where- †² 2 & 4 E. fore l. 6.

240 *Elements of Conic Sections. Book III.*

fore as DC is to GE, so is BA to BF. But
^{* 14E. l. 5.} DC is equal to BA, therefore GE is * equal to BF, consequently BG is equal to FE; and because the triangles BFA, BGD,
^{† 4E. l. 6.} are equiangular, as AF is to DG, † so is BF to BG. But BF is proved equal to GE, and BG to FE; wherefore as AF is to DG, so is GE to EF, consequently the rectangle contained between GD and GE,
^{‡ 16E. l. 6.} is ‡ equal to the rectangle contained between AF and FE; and the point A is in the periphery of the hyperbola, which has the lines EB and EC for its asymptotes, consequently the point D is * in the periphery of the hyperbola, which has the same right lines for their asymptotes.
^{* cor. 1. pr. 12 of this.}

Therefore, If from any point in the periphery of an hyperbola, a right line be drawn cutting both the asymptotes, and from either of these points of intersection a part be cut off, equal to the segment of the line intercepted between the given point and the other asymptote, in such a manner, that the extremity of the line cut off and the given point, are either both within or both without the asymptotes; in the first case that point will be in the periphery

ry of the same hyperbola with the first given point, and in the last case it will be in the periphery of the opposite hyperbola; which was to be demonstrated:

PROPOSITION XV.

THEOREM XV.

If the adjacent asymptotes of two opposite hyperbolas be cut with a right line, that line will cut each of the opposite hyperbolas in one point only.

LET the line CB cut the adjacent asymptotes EB, EC, of two opposite hyperbolas, in the points C and B; I say, the line CB will cut the periphery of these opposite hyperbolas in one point only. For let H be any point taken in one of the hyperbolas, and from H let HK be drawn * parallel to BC, cutting both the ^{* 31 E. I. 12} asymptotes in K and L, and with the line BC constitute a † rectangle, equal to the † ^{29 E. I. 61} rectangle contained between KH and HL, exceeded in figure by a square, which let be upon BA or CD, the points A and D will be in the periphery of the opposite hyperbolas. For through the points A
H h and

242 *Elements of Conic Sections. Book III.*

and H draw the lines AF, AO, HM, HN, parallel to the asymptotes;

Then because FA is parallel to HM, and AB to HK, the triangles FAB, HMK,

*^{19 E. l. 1.} are * equiangular; and as FA is to HM,

†^{4 E. l. 6.} so is AB to HK. But because the rectangle contained between CA and AB is equal to the rectangle contained between

‡^{16 E. l. 6.} KH and HL, as AB is to HK, ‡ so is HL to AC; and because of the equiangular triangles ACO, HLN, as HL is to AC, so is HN to AO; wherefore as FA is to

*^{11 E. l. 5.} HM, * so is HN to AO; consequently the rectangle contained between AF and AO, is equal to the rectangle contained between HM and HN; wherefore the

†^{cor. 1. pr. 12 of this.} point A is † a point in the periphery of the hyperbola, which has the lines EB and EC for its asymptotes. After the same manner may it be demonstrated, that the point D is in the periphery of the opposite hyperbola, neither can BC cut the hyperbolas in any other point than A; for if it were supposed to cut it in another point, then would a rectangle contained between two lines, each greater or smaller than FA and AO, respectively be equal to the
the

the rectangle contained between FA and AO, which is absurd.

Therefore, *If the adjacent asymptotes of two opposite hyperbolas be cut with a right line, that line will cut each of the opposite hyperbolas in one point only; which was to be demonstrated.*

PROPOSITION XVI.

THEOREM XVI.

Any right line drawn through the center, within the angle contained between the adjacent asymptotes of opposite hyperbolas, is a second diameter.

LET CA, CB, be the asymptotes of any hyperbola, and through the center C let the right line CF be drawn within the angle ACD, which is contained between the adjacent asymptotes AC, CD; I say, the right line CF is a second diameter. For take any point in the asymptote CD, as G, and through G draw GH * parallel to the asymptote CA, cutting ^{*31 E. I. 11} the line CF in H, and make GD equal to GC; join DH, which produced cuts the
H h 2 asym.

44 *Elements of Conic Sections. Book III.*

assymptote CA in A, the line DA will
 * 15 of this. * cut the two opposite hyperbolas, which
 let be in the points K and L;

Then because GH is parallel to AC, as
 † 2 E. 1. 6 DG is to GC, † so is DH to HA. But
 DG is equal to GC, therefore DH is e-
 * 8 of this. qual to HA, and DL is * equal to AK;
 wherefore LH is equal to HK, conse-
 † def. 4 of this. quently FC is a † second diameter.

Therefore, *Any right line drawn
 through the center, within the angle
 contained between the adjacent assym-
 ptotes of opposite hyperbolas, is a second
 diameter; which was to be demonstrated.*

PROPOSITION XVII.

THEOREM XVII.

*If any right line, terminated both ways
 by the assymptotes, and meeting the
 hyperbola, be bisected in the point of
 contact, it is a tangent to the hyper-
 bola; and if it be a tangent to the
 hyperbola, and terminated both ways
 by the assymptotes, it is bisected in
 the point of contact.*

LET

LET CA, CB, be the affymptotes of any hyperbola, and AB a right line terminated both ways by them, and meeting the hyperbola in D; and if AD be equal to BD, I say, the line AB is a tangent to the hyperbola. For through D draw DE * parallel to the affymptote CB, ^{*31 E. I. 11} and cutting the other in E, and through any other point in the line AB, as H, draw FH parallel to the affymptote CB, and cutting the other in F;

Then because DE is parallel to BC, as AE is to EC, † so is AD to DB; and AB † ^{†2 E. I. 6.} is bisected in D, AC will be bisected in E; and because DE is parallel to HF, the triangles ADE, AHF, are equiangular, and as DE is to EA, † so is HF to AF. But † ^{†4 E. I. 6.} the rectangle contained between DE and EC, * is to the rectangle contained be- ^{*1 E. I. 6.} tween AE and EC, as DE is to AE; that is, as HF is to AF, and the rectangle contained between HF and FC, is to the rectangle contained between AF and FC, as HF is to AF; therefore as the rectangle contained between DE and EC, is to the rectangle contained between AE and EC, † so is the rectangle contained between ^{†11 E. I. 5.} HF and FC, to the rectangle contained
be.

tween AF and FC. But because AC is equally cut in E, and unequally in F, the rectangle contained between AE and EC, that is, the square of AE, is * greater than the rectangle contained between AF and FC; wherefore the rectangle contained between DE and EC, is greater than the rectangle contained between HF and FC; consequently the point H is † without the periphery of the hyperbola. After the same manner may it be demonstrated, that any other point in the line AB is without the periphery of the hyperbola; therefore the line AB is a ‡ tangent to the hyperbola; W. W. D.

* *E. 1. 2.*
† *cor. 1 pr. 12 of this.*
‡ *def. 12 of this.*

Again, if AB be a tangent to the hyperbola, and terminated by the asymptotes in A and B, I say, the line AB is bisected in the point of contact D; for, if not, let BG, if possible, be a part of the greater BD, equal to the smaller AD; then because AD is equal to GB, and D is a point in the periphery of the hyperbola; wherefore G is also a * point in the periphery of the hyperbola; consequently the tangent AB touches the periphery of the hyperbola in more points than one, which is absurd.

* *12 of this.*

There-

Therefore, *If any right line, terminated both ways by the asymptotes, and touching the hyperbola, be bisected in the point of contact, it is a tangent to the hyperbola; and if it be a tangent to the hyperbola, and terminated both ways by the asymptotes, it is bisected in the point of contact; which was to be demonstrated.*

Cor. 1. From hence it follows, that from one point no more than one tangent can be drawn to an hyperbola: For from the point D let AB be drawn a tangent to the hyperbola, and cutting the asymptotes in A and B, no other line can be drawn from the point D a tangent to the hyperbola; for, if possible, let KDL be drawn from the point D a tangent, cutting the asymptotes in K and L, and from D draw DE parallel to the asymptote CB, and cutting the other in E; then because AB is a tangent, AD is * equal to DB. But as ^{by this pr.} AD is to DB, so † is AE to EC; there-† ^{2 E. 1. 6.} fore AE is equal to EC; and for the same reason, because KL is a tangent, KE is equal to EC; therefore AE is equal to KE, a part to the whole, which is absurd.

Cor.

Cor. 2. From hence it is evident how to draw a tangent to an hyperbola from any point in its periphery, the asymptotes being given.

PROPOSITION XVIII.

THEOREM XVIII.

Right lines drawn from the vertexes of a transverse diameter, tangents to the opposite hyperbolas, are parallel to one another.

LET CA, CB, be the asymptotes of any hyperbola, and from M and N, the vertexes of the transverse diameter MN, let the tangents OP and QE be drawn, cutting the asymptotes in the points O, P, Q and E; I say, the tangent OP is parallel to the tangent QE. For from N ^{* 31 E. 1. 1.} and M draw NR and MS * parallel to the asymptote CB, and cutting the other in R and S;

Then because NR is parallel to CO, as ^{† 2 E. 1. 6.} PN is to NO, † so is PR to RC. But PN ^{‡ 17 of this.} is ‡ equal to NO, wherefore PR is equal to RC; for the same reason CS is equal to SE. Again, because NR is parallel to MS,

MS, the triangles CRN and CMS are equiangular, and as CN is to CM, * so is * 4E. 1. 6; CR to CS, and so is CP to CE; wherefore the triangles CPN and CME are † si- † 6E. 1. 6; milar, and the angle CEM equal to the angle CPN; consequently the tangent PO is ‡ parallel to the tangent QE. ‡ 27E. 1. 11

Therefore, *Right lines drawn through the vertexes of a transverse diameter, tangents to the opposite hyperbola, are parallel to one another*; which was to be demonstrated.

Cor. From hence it follows, that if any right line be drawn parallel to any tangent, and produced until it cut the asymptotes, the square of half the tangent is equal to a rectangle contained between the segments of the line, intercepted between any one of the points where it cuts the periphery of the hyperbola, and the asymptotes; for that rectangle is * equal to the rectangle contained be- * 8 of this tween the segments of the tangent, intercepted between the point of contact and the asymptotes, that is, the square of half the tangent.

PROPOSITION XIX.

PROBLEM I.

An hyperbola and its asymptotes being given, to draw a right line which will be a tangent to the hyperbola, and parallel to a given right line, which cuts both the asymptotes of the same or opposite hyperbolas.

LET DE be any hyperbola, and CA and CB its asymptotes, and FG a given right line, cutting the asymptotes in F and G; it is required to draw a tangent to the hyperbola DE, parallel to the right line FG. Bisect the line FG in the point H, join HC to the center, and from any point, as E, in the periphery of the hyperbola, draw EL * parallel to HC, cutting the asymptotes in K and L, and find †^{13 E. L. 6.} a † mean proportional between LE and EK, which let be CD, in the line CH produced; then because LE is to CD, as CD is to EK, the rectangle contained between LE and EK, is ‡^{17 E. I. 6.} equal to the square of CD; and E is a point in the periphery of the hyperbola, wherefore D is also a §^{9 of this.} * point in the periphery of the hyperbola.

la. Through D draw the line ADM parallel to FG, cutting the asymptotes in F and G; A & M

Then because AM is parallel to FG, the triangles FHC, DCM, are equiangular, and for the same reason the triangles ADC and HCG are equiangular; wherefore as HC is to CD, * so is HF to DM, and so * 4 E. I. 6. is HG to AD; consequently as HF is to DM, † so is HG to AD. But HG is equal to HF, wherefore AD is † equal to DM; therefore the line AM is a * tangent to the given hyperbola DE, and parallel to the given line FG, cutting both the asymptotes of the same or opposite hyperbolas; *which was to be done.*

PROPOSITION XX.

THEOREM XIX.

If from any point in the periphery of an hyperbola a right line be drawn, bisecting the angle contained between two right lines drawn from the same point to the foci, that line is a tangent to the hyperbola; and if it be a tangent to the hyperbola, it will bisect the angle contained between two

I 1 2

right

right lines drawn from the point of contact to the foci.

LET AB be any hyperbola, C and D its foci, and from any point, as B, in the periphery, let the right line BF be drawn, bisecting the angle CBD, contained between the lines drawn from the given point to the foci; I say, the line BF is a tangent to the hyperbola. For about the center B, with the distance BD, to the nearest focus describe the circle DKG, cutting the line BC in G, and join DG, which cuts the line BF in H; take any other point in the line BF, as E, join the lines EC, ED and EG;

Then because BG is equal to BD, and BH is common, and the contained angles GBH, DBH, equal to one another; therefore *E. I. I.* the base GH is equal to HD, and the angles GHB, DHB, equal to one another, and consequently right angles. Again, because GH is equal to HD, and HE common, and the contained angles right angles; therefore the base EG is equal to the base ED; wherefore if about the center E, with the distance ED, a circle be described, it will pass through the

the point G. Let DLG be that circle, cutting the line EC in M; then because C is a point taken without the circle MLD, and from it CM and CG are drawn to the convexity, one of which, as CM, passes through the center; wherefore CM * is ^{† 3 E. 1. 3.} the least that can be drawn from that point to the convexity; wherefore CM is less than CG. But CG is the excess by which CB exceeds BD, that is, † the transverse ^{† 1 of this.} axis; consequently CM is less than the transverse axis, and CM is the excess by which CE exceeds ED; wherefore † E is ^{† cor. 2. pr. 2 of this.} a point without the periphery of the hyperbola. After the same manner may it be demonstrated, that any other point in the line BF is without the periphery of the hyperbola, consequently * BF is a tan- ^{* def. 10 of this.} gent to the hyperbola.

Again, if BF be a tangent to the hyperbola, it will bisect the angle CBD; for, if not, a line may be drawn from the point B, † bisecting the angle CBD, which (by ^{† 9 E. 1. 11.} the above demonstration) is a tangent to the hyperbola, and from the same point B there will be drawn more than one tangent to the hyperbola, which † is absurd; ^{† cor. 1. pr. 17 of this.} where-

wherefore the line BF bisects the angle CBD.

Therefore, *If from any point in the periphery of an hyperbola a right line be drawn, bisecting the angle contained between two right lines drawn from the same point to the foci, that line is a tangent to the hyperbola; and if it be a tangent to the hyperbola, it will bisect the angle contained between two right lines drawn from the point of contact to the foci; which was to be demonstrated.*

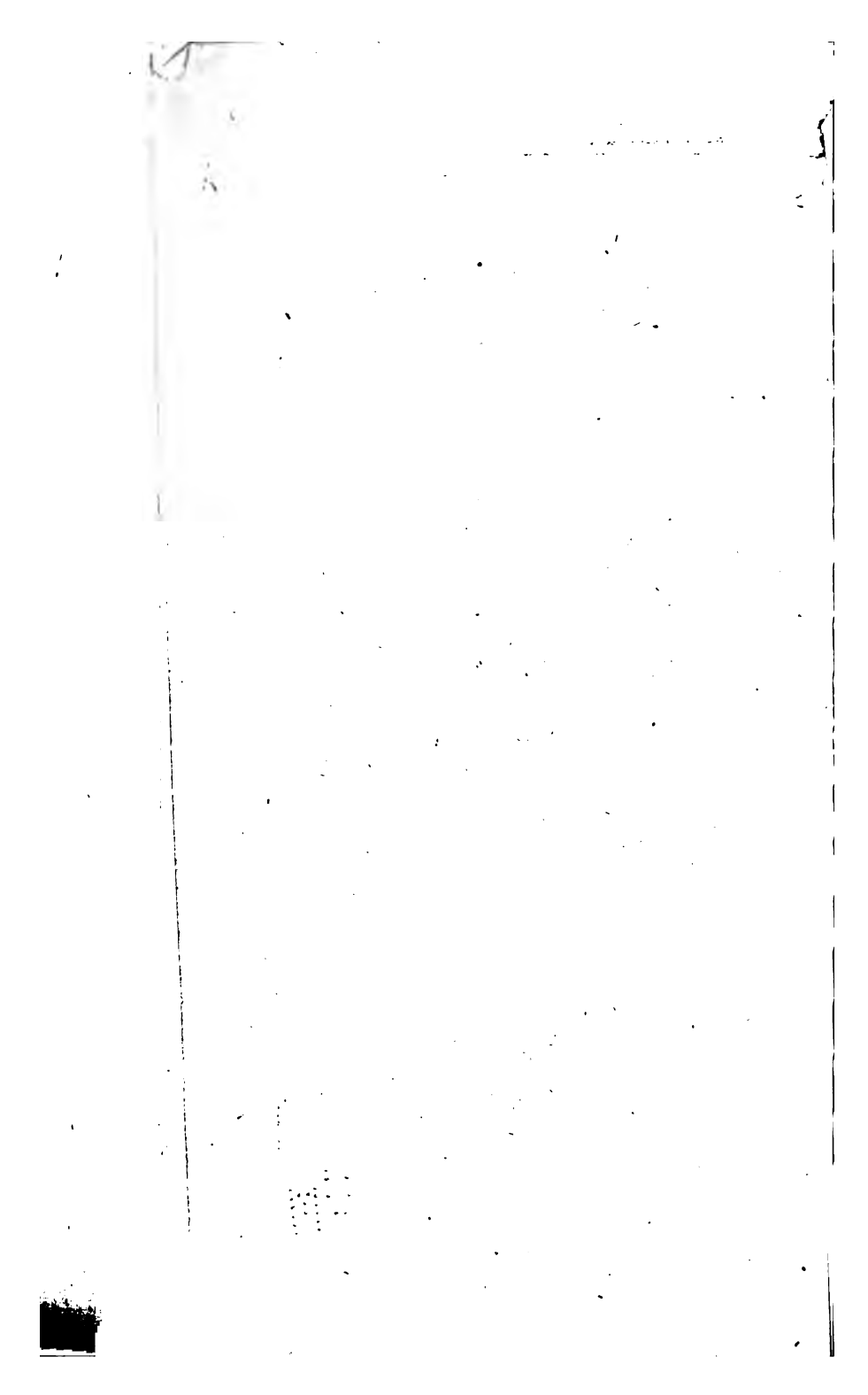
PROPOSITION XXI.

THEOREM XX.

If any right line be drawn a tangent to an hyperbola, cutting the asymptotes, the rectangle contained between the segments of the asymptotes, intercepted between the tangent and center, is equal to the rectangle contained between the segments of the asymptotes intercepted between the center and any other tangent.

LET CA, CB, be the asymptotes of any hyperbola, and let AE, BG, be
two





two tangents touching it in the points D and F, and cutting the asymptotes in the points A, E, B and G; I say, the rectangle contained between AC and CE, the segments of the asymptotes, intercepted between the center, and tangent AE, is equal to the rectangle contained between CB and CG, the segments of the asymptotes, intercepted between the center, and tangent BG. For from the points of contact, D and F, draw the lines DM, DH, FK and FL, respectively * parallel *_{31 E. l. 1.} to one asymptote, and cutting the other in the points M, H, K and L;

Then because the line MD is parallel to CE, the triangles ADM, AEC, are † e-_{(29 E. l. 1.} quiangular, and as AE is to AD, ‡ so is ‡_{4 E. l. 6.} EC to DM. But AE is * double AD, *_{17 of this.} wherefore EC is also double DM; also AC is double AM; consequently the rectangle contained between AC and CE, is four times the rectangle contained between CM and MD, because of the parallel lines DH and DM. After the same manner may it be demonstrated, that the rectangle contained between BC and CG, is four times the rectangle contained between LF and FK. But the rectangle contained between
 tween

tween DM and DH, is equal to the rectangle contained between FL and FK; wherefore the rectangle contained between AC and CE, is * equal to the rectangle contained between BC and CG.

*10 of this

Therefore, *If any right line be drawn a tangent to an hyperbola, cutting the asymptotes, the rectangle contained between the segments of the asymptotes, intercepted between the center and tangent, is equal to the rectangle contained between the segments of the asymptotes, intercepted between the center and any other tangent; which was to be demonstrated.*

Cor. I. From hence it follows, that if the points of intersection of the asymptotes, and tangents to the same or opposite hyperbolas, as A, B, E and G, as also D and F, be joined with right lines; these lines are all parallel to one another. For join EF, which cuts the line AB in N; then because the rectangle contained between AC and CE, is equal to the rectangle contained between BC and CG, as AC is to CB, † so is CG to CE; wherefore GE is ‡ parallel to AB; and because GB, EN, cut the parallel

†16 E. I. 6.

‡ 2 E. I. 6.

parallel lines AB, GE, as GF is to FB, so is EF to FN. But as GF is to FB, so is AD to DE, because they are bisected in the points D and F; therefore as EF is to FN, *so is ED to DA; *17 of this; consequently DF is † parallel to AB. † 2 E. 1. 6.

Cor. 2. The segments of any two tangents, terminated by the asymptotes, are proportionally divided in the points of contact, and where they mutually cut each other; because AB is parallel to FD, as AO is to OD, so is BO to OF.

PROPOSITION XXII.

THEOREM XXI.

If from any point in the periphery of an hyperbola a line be drawn, cutting the transverse, and parallel to its second diameter, the square of the transverse will be to the square of its second diameter, as the rectangle contained between the segments of the transverse, intercepted between its vertexes and the line drawn cutting it, is to the square of that line; and if the square of the transverse be to the square of its second diameter, as

K k

a rect-

a rectangle contained between the segments of the transverse, intercepted between its vertexes, and the point where a line drawn parallel to its second diameter cuts it, is to the square of that parallel line, the other extremity of the line parallel to the second diameter will be in the periphery of the hyperbola. But if a line be drawn from any point in the periphery of an hyperbola, cutting the second, and parallel to its transverse diameter, the square of the second is to the square of its transverse diameter, as the sum of the squares of half the second diameter and its segment, intercepted between the center and the line drawn cutting it, is to the square of that line; and if the square of the second be to the square of its transverse diameter, as the sum of the squares of half the second diameter and its segment, intercepted between the center and the point where it is cut, by a line drawn parallel to its transverse diameter, is to the square of that parallel line, the other extremity of that parallel
line

line is in the periphery of the hyperbola,

LET AB be any transverse, and HK its second diameter, of any hyperbola, C its center, CD and CE its asymptotes; and from any point in its periphery, as F, let FG be drawn parallel to the second, and cutting its transverse diameter in G; I say, the square of the transverse AB is to the square of its second diameter HK, as the rectangle contained between AG and GB, the segments of the transverse, intercepted between its vertexes and the line drawn cutting it, is to the square of that line FG. For from B, the vertex of the diameter AB, draw DE a tangent to the hyperbola, cutting the asymptotes in D and E, the line DB will be * equal and parallel to HC, and produce FG until it cut the asymptotes in M and L;

* *def. 13 of this.*

Then because LG is parallel to DB, the triangles LGC and DBC are † equiangular, † 29 E. 1. 1. and as LG is to GC, ‡ so is DB to BC; ‡ 4 E. 1. 6. wherefore the square of LG is * to the square of GC, as the square of DB is to the square of BC. But the square of DB is † equal to the rectangle contained be-

† *cor. 1. pr. 18 of this.*

tween LF and FM; and because the whole, *viz.* the square of CG, is to the whole, *viz.* the square of LG, as a part, *viz.* the square of CB, is to a part, *viz.* the rectangle contained between LF and FM; therefore the residue, *viz.* the rectangle

‡ 19 E. l. 5. contained between AG and GB, * is to the residue, *viz.* the square of FG, as the square of CG is to the square of GL, that is, as the square of CB is to the square

† def. 13 of
this.

of DB, which is † equal to CH, and by quadrupling the last ratio it will be, as the square of the transverse diameter AB is to the square of its second diameter HK, so is the rectangle contained between AG and GB, the segments of the transverse, intercepted between its vertexes and the line drawn parallel to the second diameter, to the square of that line FG: And if the square of the transverse AB be to the square of its second diameter HK, as the rectangle contained between AG and GB, the segments of the transverse, intercepted between its vertexes and the point G, where the line FG, parallel to the second diameter, cuts it, is to the square of that line FG; the point F will be in the periphery of the hyperbola; for, if not, let FG, if
pos-

possible, cut the periphery in any other point, as N; then because NG is drawn from a point in the periphery of the hyperbola, parallel to the second diameter HK, and cutting its transverse AB in G, as the square of AB is to the square of HK, * so is the rectangle contained between AG and GB, to the square of GN; wherefore the rectangle contained between AG and GB, is to the square of GN, as the same rectangle contained between AG and GB, is to the square of GF; consequently the square † of GN is equal to the square of GF, and the line GN equal to the line GF, a part equal to the whole, which is absurd; therefore the point F is a point in the periphery of the hyperbola.

Again, if from the point F the line FO be drawn parallel to the transverse AB, and cutting the second diameter HK in O; I say, the square of the second diameter HK is to the square of its transverse AB, as the sum of the squares upon CH and CO, half the second diameter, and its segment intercepted between its center and the line cutting it, is to the square of that line FO. For from F draw FG parallel to the second diameter HK, and cutting its transverse

verse in G; then because FG is parallel to OC, and GC to FO, the figure OCGF is a parallelogram, and GC * equal to FO. But because FG is drawn from the point F in the periphery of the hyperbola, parallel to the second diameter HK, and cutting its transverse in G, as the square of *prop. dem.* CB is to the square of CH, † so is the rectangle contained between AG and GB, ‡ *cor. 4. E.* to the square of FG or CO. By † inversion, as the square of HC is to the square of CB, so is the square of OC to the rectangle contained between AG and GB; wherefore as the square of HC is to the *prop. 1. 5.* square of CB, * so is the sum of the two squares described upon HC and OC, to the square of CB, together with the rectangle contained between AG and GB, that *prop. 1. 2.* is, the † square of CG or FO; and by quadrupling the first ratio it will be, as the square of HK, the second diameter, is to the square of its transverse AB, so is the sum of the squares upon HC and CO, viz. half the second diameter, and its segment intercepted between its center and the line drawn cutting it, to the square of FO.

Lastly, if from the point F, FO be drawn parallel to the transverse diameter AB,

AB, and cutting its second diameter HK in O, and if the square of HK be to the square of AB, as the sum of the squares of CH and CO is to the square of FO, the point F will be in the periphery of the hyperbola; for, if not, let the line FO cut the periphery, if possible, in any other point, as Q; then because QO is drawn from a point in the periphery of the hyperbola, parallel to the transverse diameter AB, and cutting its second diameter HK in O, as the square of HK is to the square of AB, * so is the sum of the squares upon HC and CO to the square of QO; wherefore as the sum of the squares upon HC and CO is to the square of QO, † so is † † E. I. 5; the sum of the same squares upon HC and CO to the square of FO; consequently the square of QO is ‡ equal to the square ‡ ‡ E. I. 5; of FO, and the line QO to the line FO, a part equal to the whole, which is absurd; wherefore the point F is in the periphery of the hyperbola.

Therefore, *If from any point in the periphery of any hyperbola, &c.* which was to be demonstrated.

Cor. 1. From hence it follows, that if from two or more points in the periphery of
an

an hyperbola, right lines be drawn parallel to a second diameter, and cutting its transverse, the squares of these lines have the same proportion to one another, as the rectangles contained between the segments of the transverse diameter, intercepted between its vertexes and these respective lines.

Cor. 2. If from two or more points in the periphery of the same or opposite hyperbolas, right lines be drawn parallel to any transverse, and cutting its second diameter, the squares of these lines have the same proportion to one another, as the sum of the squares of half the second diameter, and its segment intercepted between the center and the first mentioned line, is to the sum of the squares of half the same second diameter, and its segment intercepted between its center and the second mentioned line, &c.

Cor. 3. Any transverse and its second diameter are conjugate diameters: For let FG and FO cut the same and opposite hyperbolas in R and P, as the rectangle contained between AG and GB is to the square of FG, * so is the same rectangle contained between AG and GB to the

* *cor. 1 of this pr.*

the square of GR; wherefore FG is * e- * 9 E. 1. 5;
 equal to GR; and as the sum of the
 squares upon CH and CO, is to the
 square of FO, † so is the sum of the † cor. 2. of
 same two squares upon CH and CO, to *this prop.*
 the square of OP; wherefore the square
 of OP is ‡ equal to the square of OF, ‡ 9 E. 1. 5;
 and the line OP equal to the line OF;
 consequently the two diameters HK and
 AB are * *conjugate diameters*. Also * def. 7 of
 two diameters cannot be conjugate to *this.*
 one and the same transverse or conjugate
 diameter.

Cor. 4. If from two points, one of which
 is in the periphery of the hyperbola,
 two right lines be drawn parallel to the
 transverse, and cutting its second dia-
 meter, or parallel to the second, and
 cutting its transverse; and if in the first
 case the squares of these lines have the
 same proportion to one another, as the
 sum of the squares of half the second
 diameter, and the segments intercepted
 between the center and these respective
 lines; or, in the second case, the squares
 of these lines have the same proportion
 to one another, as the rectangles con-
 tained between the segments of the dia-

L I meter,

meter, intercepted between its vertexes and these lines, the other point will also be in the periphery of the same or opposite hyperbolas.

Cor. 5. Any right line, terminated by the same or opposite hyperbolas, and bisected by any transverse, or its second diameter, is parallel to the other, and all ordinates to the same diameter are parallel to one another. Also right lines parallel to either of the conjugate diameters, and cutting off from the other equal segments to the center, are equal to one another; and if they be equal to one another, and parallel to either of the two conjugate diameters, the segments of the other, intercepted between them and the center, are equal to one another.

Cor. 6. Two or more right lines parallel to one another, and terminated by the same or opposite hyperbolas, that diameter which bisects any one of them will bisect them all; and that right line which bisects two or more right lines, terminated by the same or opposite hyperbolas, is a diameter.

Cor. 7. The rectangle contained between the

the segments of any transverse diameter, intercepted between the ordinate and its vertexes, is to the square of the semi-ordinate, as the diameter is to its latus rectum: For the transverse is to its conjugate, * as its conjugate is to the latus rectum of the transverse; therefore the square of the transverse is to the square of its conjugate, as the transverse is to its latus rectum. But the square of the transverse is to the square of its conjugate, † as the above mentioned † *prec. pr.* rectangle is to the square of the semi-ordinate; consequently the transverse diameter is to its latus rectum, ‡ as the † *E. l. 5.* rectangle contained between the segments of it, intercepted between any ordinate and its vertexes, is to the square of that semi-ordinate. After the same manner may it be demonstrated, that any second diameter is to its latus rectum, as the sum of the squares of half the second diameter, and its segment intercepted between the center and any ordinate, is to the square of that semi-ordinate.

PROPOSITION XXIII.

THEOREM XXII.

A right line drawn through the vertex of any transverse diameter, and parallel to the ordinate of that diameter, is a tangent to the hyperbola; and if it be a tangent to the hyperbola, it will be parallel to the ordinate of the transverse diameter drawn through the point of contact.

LET CB be a transverse diameter to the hyperbola BF, C its center, CD and CE the asymptotes, and FR an ordinate to the diameter CB, through the vertex B let BD be drawn parallel to the ordinate FR; I say, DB is a tangent to the hyperbola. For because FR is an ordinate to the diameter BC, FR will be * parallel to the second diameter; wherefore DE will

* cor. 5. pr.
22 of this.

† 50 E. l. i.

‡ def. 13
of this.

the ordinate of the

hyperbola BF. Again, if DB be a tangent to the hyperbola BF, it will be parallel to the second diameter, which is conjugate to CB; for because BD is a tangent to the hyperbola, it will * be parallel to the second diameter,

* def. 13 of
this.

ameter, which is conjugate to CB. But FR is ^{* cor. 5. pr.} parallel to the same diameter; wherefore ^{21 of this.} the tangent DB is parallel to the ordinate FR.

Therefore, *If a right line be drawn through the vertex of any transverse diameter, and parallel to an ordinate of that diameter, it is a tangent to the hyperbola; and if it be a tangent to the hyperbola, it will be parallel to the ordinates of the transverse diameter drawn through the point of contact; which was to be demonstrated.*

Cor. Two right lines cutting one another, and not drawn through the center, and terminated by the same or opposite hyperbolas, cannot bisect each other; for if they did bisect each other, they would be both ^{* def. 8 of this.} ordinates to the same diameter, and consequently would be ^{† cor. 5. pr. 12 of this,} parallel to one another, which is absurd.

PROPOSITION XXIV.

THEOREM XXIII.

If a right line, which is a tangent to an hyperbola, cut any transverse diameter, and from the point of contact an ordinate be drawn to the same diameter,

ameter, the rectangle contained between the segments of the diameter intercepted between the ordinate and center, and the ordinate and tangent, is equal to a rectangle contained between the segments of the diameter intercepted between the ordinate and the vertexes of the diameter. Also the rectangle contained between the segments of the diameter, intercepted between the tangent and center and tangent and ordinate, is equal to a rectangle contained between the segments of the diameter, intercepted between the tangent and vertexes of that diameter. But if the tangent cut any second diameter, and from the point of contact an ordinate be drawn to the same diameter, the rectangle contained between the segments of the diameter, intercepted between the ordinate and center, and ordinate and tangent, is equal to the sum of the squares described upon half the diameter, and the segment of the diameter intercepted between the center and ordinate. Also the rectangle contained between the segments of the dia-
 dia-

diameter, intercepted between the tangent and center, and tangent and ordinate, is equal to the sum of the squares described upon half the diameter, and that segment of it intercepted between the center and tangent.

LET A be the center of any hyperbola, AB and AC its asymptotes, and let DE be a tangent touching it in the point D, and cutting the transverse diameter in the point K, and from the point of contact D let DH be drawn an ordinate to the diameter GF, cutting it in H; I say, the rectangle contained between AH and HK, the segments of the diameter, intercepted between the center and ordinate, and tangent and ordinate, is equal to the rectangle contained between GH and HF, the segments of the same diameter, intercepted between the ordinate and the vertexes of the diameter. For let the tangent DK cut the asymptotes in E and C, and from F, the vertex of the diameter FG, let BF be drawn a * tangent to ^{† 17 of this} the hyperbola, cutting the asymptotes in B and L, and from F let FM be drawn † parallel to ED, cutting the asymptote † 11 E. I. 12

AB

AB in M, and let HD produced cut the asymptote AC in N: *Join EL & BC*

Then because BL is drawn from the vertex of the diameter GF, a tangent to the hyperbola, it is * parallel to the ordinate DH; also because the tangents BL, EC, cut one another in O, as DO is to

† cor. 2. pr. 2 of this. OC, † so is FO to OB. But as OD is to OC, † so is NL to LC, and as FO is to OB, so is ME to EB; wherefore as NL

* *11 E. l. 5.* is to LC, * so is ME to EB; and because *† cor. 1. pr. 2 of this.* EL is † parallel to BC, as CL is to LA, *† 22 E. l. 5.* so is BE to EA; and by † equality, as LN is to LA, so is EM to EA; and by

* *18 E. l. 5.* composition, * as NA is to LA, so is MA to EA. But as AN is to LA, † so is HA to FA, and as MA is to EA, so is FA to KA; consequently as HA is to FA, so is FA to KA; therefore the rectangle con-

† 17 E. l. 6. tained between HA and AK, is † equal to the square of FA. Again, because FG is equally cut in A, and HF added to it, the rectangle contained between GH and HF, together with the square of FA, is

† 6 E. l. 2. † equal to the square of HA; also because HA is any how cut in K, the square of

† 2 E. l. 2. HA is † equal to the rectangle contained between HA and AK, together with the rect-

rectangle contained between HA and HK; wherefore the rectangle contained between GH and HF, together with the square of FA, is equal to the rectangle contained between HA and AK, together with the rectangle contained between HA and HK. But the square of FA has been demonstrated to be equal to the rectangle contained between HA and KA; consequently the rectangle contained between HA and HK, that is, the rectangle contained between the segments of the diameter, intercepted between the ordinate and center, and ordinate and tangent, is equal to the remaining rectangle contained between GH and HF, *viz.* the rectangle contained between the segments of the diameter intercepted between the ordinate and vertexes of the diameter.

I say also, the rectangle contained between HK and KA, the segments of the diameter, intercepted between the tangent and ordinate, and the tangent and center, is equal to the rectangle contained between GK and FK, *viz.* the segments of the diameter intercepted between the tangent and vertexes of the diameter. For because FG is equally cut in A, and unequally in

M m
K,

K, the rectangle contained between GF and FK, together with the square of KA, ^{*; E. 1. 2.} is * equal to the square of FA. But (by the above demonstration) the square of FA is equal to the rectangle contained between HA and AK; and because HA is any how cut in K, the rectangle contained between ^{†; E. 1. 2.} HA and AK, is † equal to the rectangle contained between HK and KA, together with the square of AK; therefore the rectangle contained between GK and FK, together with the square of KA, is equal to the rectangle contained between HK and KA, together with the same square of KA; take from both the common square of KA, and there will remain the rectangle contained between HK and KA, *viz.* the rectangle contained between the segments of the diameter, intercepted between the tangent and ordinate, and tangent and center, equal to the rectangle contained between GK and FK, *viz.* the rectangle contained between the segments of the diameter intercepted between the vertexes and tangent; W. W. D.

Part 2. When the tangent DE cuts the second diameter AP in Q, I say, the rectangle contained between QR and RA, the seg-

segments of the diameter, intercepted between the ordinate and center, and the ordinate and tangent, is equal to the sum of the squares described upon AP and AR, *vis.* half the second diameter, and its segment intercepted between the center and ordinate. For let GF be the conjugate transverse diameter to SA, which cuts the tangent DE in K, and from the point of contact D let DH be drawn an ordinate to the transverse diameter AF, cutting it in H; then (by the above demonstration) as HA is to FA, so is FA to KA; therefore as HA is to KA, * so is the square of HA ^{4 cor. 20 E. l. 6.} to the square of FA; and by division, as HK is to KA, † so is the rectangle con- † 17 E. l. 5; tained between GH and HF, to the square of FA. But as the rectangle contained between GH and HF, is to the square of FA, ‡ so is the square of DH or AR, which ‡ 22 of this; is equal to it, to the square of AP; wherefore as HK is to KA, * so is the square of RA to the square of AP. But as HK is to KA, † so is DK to KQ, or RA to AQ; † 2 E. l. 6; therefore as the square of RA is to the square of AP, so is RA to AQ; wherefore as RA is to AP, ‡ so is AP to AQ; ‡ ^{converse cor. 20 E. l. 6.} consequently the square of AP is * equal * 17 E. l. 6.

to the rectangle contained between RA and AQ; add to both the square of RA, and the two squares described upon RA and AP will be equal to the rectangle contained between QA and AR, together with the square of AR. But because the line QR is any how cut in A, the rectangle contained between QR and RA, is * equal to the rectangle contained between QA and AR, together with the square of AR; consequently the rectangle contained between QR and RA, that is, the rectangle contained between the segments of the diameter, intercepted between the ordinate and center, and ordinate and tangent, is equal to the sum of the two squares described upon AP, half the diameter, and RA the segment of the diameter intercepted between the ordinate and center.

Likewise I say, the rectangle contained between RQ and QA, the segments of the diameter, intercepted between the tangent and center, and tangent and ordinate, is equal to the sum of the squares described upon AP and AQ, viz. half the diameter, and the segment of it intercepted between the tangent and center. For (by the last part of the demonstration) the square

Square of AP is equal to the rectangle contained between RA and AQ; add to both the common square of AQ, and the two squares described upon AP and AQ, is equal to the rectangle contained between RA and AQ, together with the square of AQ. But because the line QR is any how cut in A, the rectangle contained between RQ and QA, is * equal to the rectangle * 3 E. I. 21 contained between RA and AQ, together with the square of QA; consequently the rectangle contained between RQ and QA, the segments of the diameter, intercepted between the tangent and center, and the tangent and ordinate, is equal to the sum of the squares described upon AP, half the diameter, and QA its segment intercepted between the tangent and center.

Therefore, *If a right line, which is a tangent to an hyperbola, cut any right line, &c. which was to be demonstrated.*

Cor. 1. From hence it follows, because

AH is to AF, † as AF is to AK, and † ^{part 1. of this.} AR is to AP, as † AP is to AQ, a semi-diameter is a mean proportional between the segments of the same diameter, intercepted between the center and tangent, and the center and ordinate, drawn

drawn from the point of contact of the tangent.

Cor. 2. The segments of the diameter, intercepted between its vertexes and the ordinate, have the same proportion to one another, as the segments of the same diameter, intercepted between its vertexes and the tangent; Because AH is to AF , as AF is to KA , by conversion, as AH is to HF , * so is AF to FK ; and by doubling the antecedents, as HG , together with HF , is to HF , so is FG to FK ; and by division, as HG is to HF , so is KG to FK . After the same manner may it be demonstrated of the second diameter QS .

Cor. 19 E.
6.5.

Cor. 3. When an ordinate to a second diameter, drawn from the point of contact of a tangent, cuts it in one of its vertexes, the tangent will cut it in the other vertex.

Cor. 4. If from any point in the periphery of an hyperbola an ordinate be drawn to any diameter, and from the same point a line be drawn cutting the diameter; and if half the diameter be a mean proportional between the segments of it, intercepted between the center and

and ordinate, and the center and that line; the line drawn cutting it will be a tangent to the hyperbola. For let DH be an ordinate, cutting the diameter AF in H , drawn from the point D in the periphery of the hyperbola FD , and from D let DK be drawn cutting it in K , so as FA , half the diameter, is a mean proportional to HA and KA , DK is a tangent to the hyperbola; for, if not, let any other line, as DT , be drawn a tangent to the hyperbola, cutting the diameter in T ; then as HA is to FA ,
 * so is FA to TA . But as HA is to FA ,
 † so is FA to KA ; wherefore as FA is
 to KA , so is FA to TA ; consequently
 KA is † equal to TA , a part to the † whole, which is absurd.

* cor. 1 of this.

† hypoth.

† E. l. 51

PROPOSITION XXV.

THEOREM XXIV.

If a right line be drawn a tangent to one of the conjugate hyperbolas, and through the center two right lines be drawn, one through the point of contact, and the other parallel to the tangent, and cutting one of the adjacent

cent hyperbolas, that line will be half the second diameter, conjugate to the transverse drawn through the point of contact; and if that last mentioned line be half the second diameter, conjugate to that transverse drawn through the point of contact, its extremity will be in the periphery of one of the adjacent hyperbolas.

LET AB, CD, EF and GH, be four conjugate hyperbolas, K their center, KL and KM their asymptotes; let LA be a tangent to the hyperbola AB, touching it in A, and cutting the asymptotes in L and M, and through the center K let KA be drawn to the point of contact, as also KC parallel to the tangent LM, cutting the periphery of the hyperbola, adjacent to AB, in the point C; I say, KC is half the second diameter, which is conjugate to KA. For through the points A and C draw the lines AN, ^{† 31 E. l. 1.} CO, * parallel to the asymptote KL, and cutting the other in the points N and O; Then because LM is a tangent to the ^{† 17 of this.} hyperbola AB, LA is † equal to AM; ^{‡ 2 E. l. 6.} but as LA is to AM, ‡ so is KN to MN; where-

wherefore KN is equal to MN. Again, because A and C are two points in the periphery of the adjacent hyperbolas AB and CD, and from these points the lines AN, CO, are drawn parallel to the asymptote KL, cutting the other in N and O, therefore the rectangle contained between AN and NK, that is, NM is * equal to the rectangle contained between CO and OK; ^{* cor. 2. pr. 10 of this.} wherefore as CO is to AN, † so is NM to OK. But because CO is parallel to AN, and CK to AM, the triangles COK, ANM, are similar, and as CO is to AN, † so is OK to NM; therefore as OK is to NM, so is NM to OK; consequently the square of NM is * equal to the square of OK, ^{* 17 E. I. 6.} and the line NM equal to the line OK. But as OK is to NM, so is CK to AM; wherefore CK is equal to AM; consequently CK is † half the second diameter, ^{† def. 14 of this.} conjugate to the transverse AK, drawn through the point of contact.

Again, if CK be equal to half the second diameter, which is conjugate to the transverse AB, the construction remaining the same as before; I say, the point C will be in the periphery of one of the adjacent hyperbolas to AB. For because CK

N n is

is half the second diameter to the transverse AK, CK is * equal and parallel to AM, and CO is parallel to AN; therefore the two triangles COK, ANM, have

† 29 E. I. the angles COK, OKC, in the one † equal to the angles ANM, NMA, in the other, each to each, and the side CK equal to the side AM; wherefore the side

‡ 26 E. I. CO is ‡ equal to the side AN, and OK to NM, that is KN; consequently the rectangle contained between CO and OK, is equal to the rectangle contained between AN and NK. But A is a point in the periphery of the hyperbola AB, and C is a point within the angle OKL, adjacent to the angle LKM, contained between the asymptotes of the hyperbola AB; consequently the point C * is in the periphery of the hyperbola adjacent to AB.

* cor. 4. pr. 32 of this.

Therefore, *If a right line be drawn a tangent to one of the conjugate hyperbolas, and through their center two right lines be drawn, one through the point of contact, and the other parallel to the tangent, and cutting one of the adjacent hyperbolas, that line will be half the second diameter, conjugate to the transverse, drawn through the point of*

of

of contact; and, if that last mentioned line be half the second diameter, conjugate to that transverse drawn through the point of contact, its extremity will be in the periphery of one of the adjacent hyperbolas; which was to be demonstrated.

Cor. 1. From hence it follows, that if CK be half a second diameter, conjugate to the transverse AK, in the hyperbola AB, reversely AK will be half a second diameter, conjugate to the transverse CK, in the adjacent hyperbola CD. For join the lines CA, CL, and let CL cut the asymptote KM in P; then because CO was proved to be equal to AN, and is parallel to it, therefore CA is * parallel *^{34 E. 1. 1.} and equal to ON, and as LA is to AM, † so is LC to CP. But LA is equal to †^{2 E. 1. 6.} AM, wherefore LC is equal to CP, and C is a point in the periphery of the hyperbola CD; therefore LP is a † tangent †^{17 of this.} to the hyperbola; and because CK is half the second diameter, conjugate to the transverse AK, therefore CK is * parallel and equal to LA; consequently LC is parallel and equal to AK; ^{this.} therefore AK is half the second diame-

N n 2 ter,

ter, conjugate to the transverse CK in the adjacent hyperbola CD.

Cor. 2. A right line drawn from the point of intersection of two tangents, drawn from the vertexes of two conjugate diameters to the center, is one of the asymptotes of the conjugate hyperbolas. Let L be the point of intersection of the two tangents AL, CL, drawn from the vertexes of the conjugate diameters AK, KC, LK will be one of the asymptotes; for, if not, let any other line, as KQ, be the asymptote, cutting the tangent AL in Q; then because CL is parallel to AK, and CK to LA, the figure CKLA is a parallelogram, and CK is * equal to LA. But CK is \neq equal to QA, therefore QA is equal to LA, a part to the whole, which is absurd; consequently LK is one of the asymptotes of the conjugate hyperbolas.

* 34 E. I. I.
† def. 14 of
this.

Cor. 3. A right line drawn from two points in the periphery of two adjacent hyperbolas, and bisected by one of the asymptotes, is parallel to the other; and if it is parallel to one asymptote, it is bisected by the other. Let CA be drawn from the two points C and A in the adjacent

cent hyperbolas, and bisected in R by the asymptote KL, it is parallel to the other asymptote KM. For draw the lines CO, AN, parallel to the same asymptote KL, cutting the other in O and N; then because the lines CO, RK, AN, are all parallel to one another, as CR is to RA, * so is OK to KN; wherefore OK is equal to KN. But the rectangle contained between CO and OK, is † equal to the rectangle contained between AN and NK; therefore CO is parallel and equal to AN, consequently CA is ‡ parallel to the asymptote KM. After the same manner may it be demonstrated, that if CA be parallel to the asymptote KM, it will be bisected by the other asymptote KL. Also if the line CA be bisected in R by the asymptote KL, and parallel to the other asymptote KM; and if the one extremity A be in the periphery of the hyperbola AB, the other extremity C will be in the periphery of the adjacent hyperbola. Because CR is equal to RA, OK is equal to KN, and CO to AN; wherefore the rectangle contained between AN and NK, is equal to the rectangle

* 2 E. I. 6.

† cor. 2. pr.
10 of this.

‡ 33 E. I. 1.

angle contained between CO and OK ; and A is a point in the periphery of the hyperbola AB , and C is a point within the angle PKL , adjacent to the angle LKM , contained between the asymptotes of the hyperbola AB ; therefore C is a * point in the periphery of the hyperbola adjacent to AB .

* *cor. 4. pr.*
12 of this

Cor. 4. If from the vertex of any second diameter an ordinate be drawn to any other diameter, and from the same point a line be drawn parallel to a transverse which is conjugate to that second diameter, and cutting the diameter to which the ordinate is drawn, half that last diameter is a mean proportional between its segments, intercepted between the ordinate and the other line drawn cutting it. If CK be any second diameter, and KA a transverse conjugate to it, and KTV any other diameter, and from C let CS be drawn an ordinate to KT , and CV parallel to AK , cutting the diameter TK in V , TK , half that third diameter, is a mean proportional between KS and KV ; for because C is the vertex of the second diameter to the transverse KA , it will be a point in the periphery

riphery of the adjacent hyperbola to AT, and AK will be the * second diameter, and CV will be a † tangent to the hyperbola CD; consequently KT is a ‡ mean proportional between KS and KV.

* cor. 1 of this pr.

† def. 13 of this.

‡ cor. 1. pr. 24 of this.

PROPOSITION XXVI.

THEOREM XXV.

If from the vertex of any second diameter of an hyperbola, a right line be drawn parallel to any transverse diameter, and cutting a second diameter conjugate to the transverse, the square of the transverse is to the square of its conjugate, as the square of the line drawn cutting the conjugate, is to the rectangle contained between the segments of the conjugate, intercepted between its vertexes and the line drawn cutting it. But if from the vertex of the same second diameter, a line be drawn parallel to any second diameter, and cutting a transverse which is conjugate to it, the square of the second diameter is to the square of the transverse which

is

is conjugate to it, as the square of the line drawn cutting the transverse, is to the sum of the squares described upon half the transverse diameter, and its segment intercepted between the line drawn cutting it and the center.

LET C be the center of any hyperbola, AB a transverse, and DE its conjugate diameter, and CF any other second diameter, and from the vertex F let FG be drawn parallel to the transverse diameter AB, and cutting its second diameter DE in G; I say, the square of the transverse AB is to the square of its conjugate DE, as the square of the line FG is to the rectangle contained between EG and ^{GD}GD.



^AGD

* cor. 1. pr.
25 of this.

For because F is the vertex of the second diameter FC, F will be a * point in the periphery of the hyperbola DF, adjacent to the hyperbola BK, and DE will be a transverse diameter in the hyperbola DF, which has AB for its second conjugate diameter; wherefore as the square of DE is to the square of its conjugate diameter AB, † so is the rectangle contained between EG, GD, to the square of GF, and by

† 22 of this.

by * inversion, as the square of the diameter AB is to the square of its conjugate DE, so is the square of FG, the line drawn parallel to the transverse, to the rectangle contained between EG and GD, the segments of the second diameter, intercepted between its vertexes and the line drawn cutting it. * 4 E. 1. 3.

Again, if from F, the vertex of the same second diameter FC, FH be drawn parallel to the second diameter DE, and cutting its transverse AB in H, the square of the second diameter DE is to the square of the transverse AB, as the square of FH, the line drawn cutting the transverse, is to the square of  half the transverse, together with the square of  its segment, intercepted between the center and the line drawn cutting it. CB
CH Because F is a point in the hyperbola DF, adjacent to BK, and DE is a † transverse diameter in that hyperbola, which has AB for its second diameter; and since FH is drawn from a point in the periphery of the hyperbola DF, parallel to the transverse DE, and cutting its second diameter AB, the square of the diameter AB is to the square of the diameter DE, ‡ as the sum of the squares † cor. 1 pr.
25 of this
‡ 22 of this

described upon CB and CH, is to the square of FH; and by * inversion, as the square of the second diameter DE is to the square of the conjugate AB, so is the square of FH to the sum of the squares described upon CB and CH, half the transverse, and its segment intercepted between the line drawn cutting it and the center.

* cor. 4 E.
l. 5.

From hence it follows that if from any point in an Hyperbola a right line be drawn parallel to the any transverse Diameter

Therefore, *If from the vertex of any second diameter, &c. which was to be demonstrated.*

and cutting a second Diameter conjugate to the transverse, & also from the same point a right line parallel to the same second Diameter and cutting the same conjugate transverse Diameter, then the Rectangle contained between the segments of the same transverse Diameter intercepted between the right line and its vertexes is to the square of the line drawn cutting it as the square of the line drawn cutting the second Diameter is to the sum of the squares of half the second Diameter and the segment of it intercepted between the

Cor. From hence, and prop. 22. of this, it follows, that if from any point, as K, in the periphery of an hyperbola, BK, KL, be drawn an ordinate to the second diameter DE, and from F, the vertex of any second diameter in the hyperbola BK, FG be drawn parallel to the line KL, the square of FG is to the square of KL, as the rectangle contained between EG and GD, is to the sum of the squares described upon CD and CL, viz. half the diameter DE, and its segment intercepted between the center and the line KL. For (by the above prop.) the square of DE is to the square of AB, as the rectangle contained between EG and GD, is to the square of half the second Diameter of

of GF; and the square of DE is to the square of AB, * as the sum of the squares ^{*22 of this.} described upon CD and CL, is to the square of KL; therefore the square of GF is to the square of KL, as the rectangle contained between EG and GD, is to the sum of the squares described upon CD and CL. After the same manner may it be demonstrated, that if from any point in the periphery of an hyperbola, an ordinate be drawn to a transverse diameter, and from the vertex of any second diameter a line be drawn parallel to that ordinate, the square of the line drawn parallel to the ordinate is to the square of the semi-ordinate, as the sum of the squares described upon half the transverse diameter, and its segment intercepted between the line drawn parallel to the ordinate and the center, is to the rectangle contained between the segments of the diameter intercepted between its vertexes and the ordinate.

PROPOSITION XXVII.

THEOREM XXVI.

If from the vertexes of any two conjugate diameters of an hyperbola, ordinates be drawn to any third diameter, the square described upon the segment of that third diameter, intercepted between the ordinate drawn from the vertex of the transverse diameter and the center, is equal to the sum of the squares described upon half that third diameter, and its segment intercepted between its center and the other ordinate. But the square described upon the segment of the third diameter, intercepted between the center and ordinate drawn from the vertex of the second diameter, is equal to a rectangle contained between the segments of the third diameter, intercepted between its vertexes and the ordinate drawn from the vertex of the transverse diameter,

LET BK be any hyperbola, CK half any transverse diameter, and CF half its

its conjugate, and AB any third diameter, and from their vertexes K and F let KM and FH be drawn ordinates to the diameter AB; I say, the square of CM, the segment of the third diameter AB, intercepted between the center and the ordinate KM, drawn from the vertex of the transverse diameter KC, is equal to the sum of the squares described upon CB and CH, viz. half the third diameter, and its segment intercepted between its center and the ordinate drawn from the vertex of the second diameter FC. From the point K draw the line KN * parallel to FC, cutting the third diameter AB in N, and from F draw FO parallel to KC, cutting the same diameter AB in O;

Then because KN is drawn from the transverse diameter KC, parallel to its conjugate FC, KN is a † tangent to the hyperbola BK, and as MC is to BC, ‡ so is BC to NC, and the rectangle contained between MC and CN is equal to the square of CB; also because FO is drawn from the vertex of the second diameter, parallel to its transverse, and cutting the diameter AB, and FH is drawn an ordinate to the same diameter AB, as CH is to CB, * so † def. 13 of this.
‡ cor. 1. pr. 24 of this.
* cor. 1. pr. 24 of this.
is

is CB to CO ; wherefore the square of CB is equal to the rectangle contained between CH and CO ; therefore the rectangle contained between MC and CN, is equal to the rectangle contained between

*16 E.1. 6. CH and CO ; and as MC is to CH, * so is CO to CN. But because KN is parallel to FC, and KC to FO, the triangles FCO, KNC, are equiangular, and as CO is to CN, so is OF to CK ; wherefore as MC is to CH, so is OF to CK ; also because KM is parallel to FH, both being ordinates to the same diameter AB, the triangles MKC, FHO, are equiangular, and as KC is to FO, so is MC to HO ; wherefore as CH is to CM, so is CM to HO ; consequently the square of CM is equal to the rectangle contained between CH and HO. But because OH is any how cut in C, the rectangle contained between

† 3 E. 1. 2. HO and HC † is equal to the rectangle contained between OC and CH, together

‡ cor. 1 pr. 24 of this. with the square of CH, that is, ‡ the square of CB, together with the square of CH ; therefore the square of CM, the segment of the third diameter AB, intercepted between the center and the ordinate drawn from the extremity of the transverse diameter KC,

KC, is equal to the sum of the squares described upon CB. half the diameter, and CH the segment of it, intercepted between the center and the ordinate drawn from the vertex of the second diameter.

Likewise I say, the square of CH, the segment of the third diameter, intercepted between its center and the ordinate drawn from the vertex of the second diameter, is equal to the rectangle contained between AM and MB, the segments of the diameter intercepted between its vertexes and the other ordinate. For because (by the above demonstration) the square of CM is equal to the rectangle contained between OH and HC. But the square of MC is * equal to the rectangle contained between * { E. 1. 2 } AM and MB, together with the square of BC ; and the rectangle contained between OH and HC † is equal to the rectangle † { E. 1. 2 } contained between OC and CH, together with the square of CH ; take away the square of BC, and the rectangle contained between OC and CH, which are equal to one another, and there will remain the square of CH, the segment of the third diameter AB, intercepted between the center and the ordinate drawn from the vertex

tex

tex of the second diameter, equal to the rectangle contained between AM and MB the segments of the same diameter, intercepted between its vertexes and the other ordinate.

Therefore, *If from the vertexes of two conjugate diameters ordinates be drawn to any third diameter, &c. which was to be demonstrated.*

Cor. From hence it follows, that the diameter to which the ordinates are drawn is to its conjugate diameter, as the distance of any one of them from the center is to the other ordinate. Because * the square of AB is to the square of ED, as the rectangle contained between AM and MB, that is, (by the above prop.) the square of CH to the square of KM; wherefore as AB † is to DE, so is CH to KM. Again, because ‡ the square of AB is to the square of DE, as the sum of the squares described upon CB and CH, that is, (by the above prop.) the square of CM to the square of FH; wherefore as AB is to ED, so is CM to FH.

* 22 of this.

† 22 E. I. I.

‡ 26 of this.

PRO

PROPOSITION XXVIII.

THEOREM XXVII.

If the two axes of any hyperbola be unequal, the difference of the squares of the two semi-conjugate diameters is equal to the difference of the squares of the two semi-axes. But if any diameter be equal to its conjugate, any other diameter is also equal to its conjugate; and in this last case the angle contained between the asymptotes is a right angle.

LET AB be any hyperbola, CB and CD the two semi-axes, and CG and CA two semi-conjugate diameters, and if CB and CD be unequal to one another; I say, the difference of the squares described upon CB and CD , is equal to the difference of the two squares described upon CA and CG . For from the points A and G draw the lines AH , AM , GK and GL , ordinates to the two axes;

Then because the square of AC * is equal to the two squares described upon CH and HA , and the square of CG is equal to the two squares described upon CL and

PP

LG

LG; therefore the difference of the two squares described upon CA and CG, is equal to the difference of the sum of the squares described upon AH and HC, and upon CL and LG. But the square of CH ^{*27 of this} is * equal to the two squares described upon CB and CK, and the square of CL is equal to the two squares described upon CD and CM; wherefore the difference of the two squares described upon CA and CG, is equal to the difference of the sum of the three squares described upon CK, CB and AH, and the sum of the three squares described upon CD, CM and GL.

^{†34 E. l. r.} But the square of AH is † equal to the square of MC, and the square of CK is equal to the square of GL; take the two squares described upon AH and CK, and the two squares GL and MC, from both, and the difference between the two squares described upon CA and CG, the two semi-conjugate diameters, will be equal to the difference between the two remaining squares described upon CB and CD the two semi-axes.

Again, if any diameter of an hyperbola be equal to its conjugate, any other diameter will be equal to its conjugate, and the

the angle contained between the asymptotes is a right angle. For let AB be any hyperbola, CE and CF its asymptotes, and let CA and CB be two transverse diameters, and from A and B let EG and HF be drawn tangents, then will EG be * the ^{* def. 13 of this.} second diameter to the transverse CA, and HF the second diameter to the transverse CB; and if CA be equal to EA, I say, CB is also equal to BF, and the angle ECF is a right angle. For from the point A draw AK parallel to the asymptote CF, cutting the other in K, and from B † draw BL parallel to the asymptote EC, and cutting the other in L; then because KA is parallel to CG, ‡ as EA is to AG, so is EK ‡ ^{2 E. 1. 6.} to KC. But EA is equal * to AG, therefore ^{* 17 of this.} EK is equal to KC; and for the same reason CL is equal to LF; and since EK is equal to KC, and KA common, and the base EA equal to the base AC, † the angle † ^{3 E. 1. 1.} EKA is equal to the angle CKA, and each of them right angles. But EKA is ‡ equal ‡ ^{29 E. 1. 1.} to ECF, therefore the angle contained between the asymptote is a right angle; and since CL is equal to LF, and BL common, and the contained angles CLB, BLF, are right angles, the base BC * is equal to the ^{* 4 E. 1. 1.}

base BF ; and consequently the whole transverse diameter equal to HF is conjugate.

Therefore, *If the axes of any hyperbola be unequal, the difference of the squares described upon the two semi-axes is equal to the difference of the squares described upon any two semi-conjugate diameters. But if any diameter be equal to its conjugate, any other diameter is also equal to its conjugate ; and the angle contained between the asymptotes is a right angle ; which was to be demonstrated.*

PROPOSITION XXIX.

THEOREM XXVIII.

If through the vertexes of two conjugate diameters four right lines be drawn tangents to the conjugate hyperbolas, the parallelogram contained between these tangents is equal to a parallelogram contained between four tangents drawn through the vertexes of any other two conjugate diameters.

LET

LET AB, ED, be two conjugate diameters, and through their vertexes let tangents be drawn cutting one another in the points L, M, N and O, and let FG and KH be two other conjugate diameters, and through their vertexes let tangents be drawn cutting one another in the points P, Q, R and S; I say, the figures LMNO, PQRS, are parallelograms, and equal to one another. For from the points G and H draw the lines GT, GV, HW, HX, ordinates to the diameters AB and ED, and let the tangents GQ and RH cut the diameter ED in the points Y and Z; complete the parallelogram C_TaY, and join the line GY;

Then because LM is a tangent drawn through A the vertex of the transverse diameter AB, and ON is a tangent drawn through its other vertex; therefore * ON ^{*18 of this.} is parallel to LM; for the same reason LO is parallel to MN; consequently the figure LMNO is a parallelogram; and for the same reason the figure PQRS is a parallelogram. Again, because HY is a tangent cutting the transverse diameter CD in Y, and HW is an ordinate to the same diameter, drawn from the point of contact,

* as

302 *Elements of Conic Sections. Book III.*

9 cor. 1. pr. * as CY is to CD, so is CD to CW. But
24 of this. because HW, GV, are ordinates to the
 same diameter ED, and drawn from the

** cor. prop.* vertexes of two conjugate diameters, † as
27 of this. CD is to CB, so is CW to GV, or Ya,
 which is equal to it; then because the
 three magnitudes CY, CD and CB, and
 the other three, CD, CW and Ya, are
 two and two in ordinate proportion, they
 they will be in proportion by equality,

32 E. 1. 5. † as CY is to CB, so is CD to Ya; and

** 29 E. 1. 1.* the angles BCD, CYa, are * equal, there-
 fore the parallelogram BCDN is equal to
 the parallelogram CTaY. But the triangle

† 41 E. 1. 1. CGY † is half the parallelogram CTaY,
 and the same triangle CGY is also half the
 parallelogram CGQH; wherefore the pa-
 rallelogram CGQH is equal to the paral-
 lelogram CTaY, that is, the parallelogram
 BCDN; and consequently the whole pa-
 rallelogram LMNO, which is quadruple
 the parallelogram BCDN, is equal to the
 parallelogram PQRS, which is quadruple
 the parallelogram CGQH.

Therefore, *If from the vertexes of
 two conjugate diameters four tangents
 be drawn to their conjugate hyperbolas,
 the parallelogram contained between
 them*

them is equal to a parallelogram contained between the tangents drawn from the vertexes of any other two conjugate diameters; which was to be demonstrated.

PROPOSITION XXX.

THEOREM XXIX.

If any tangent to an hyperbola cut two conjugate diameters, the rectangle contained between the segments of the tangent, intercepted between the point of contact and these diameters, is equal to the square of the semi-diameter conjugate to that drawn through the point of contact.

LET H be in the periphery of the hyperbola DH , and from H let the tangent HY be drawn, cutting the two conjugate diameters AB and ED in the points Y and b , and let the diameter FG be conjugate to KH , drawn through the point of contact; I say, the rectangle contained between bH and HY is equal to the square of CG . For through the points G and

*³¹ E. l. 1. and H draw the lines GT and HX * parallel to ED;

Then because HX is parallel to CY,
 ‡² E. l. 6. ‡ as bH is to HY, so is bX to XC; consequently a rectangle contained between bH and HY is similar to a rectangle contained between bX and CX. And because HX is parallel to GT, and Hb to CG, the triangles bHX and CGT are equian-

‡⁴ E. l. 6. gular, and ‡ as Xb is to bH, so is CT to CG; and since the rectangle contained between bH and HY is similar to the rectangle contained between bX and XC, whose homologous sides are bH and bX, and the square of CT is similar to the square described upon CG; wherefore * as the rectangle contained between bH and HY is to the rectangle contained between bX and XC, so is the square of CG to the square of CT. But the rectangle contain-

‡³ E. l. 2. ed between bX and XC ‡ is equal to the rectangle contained between bC and CX, together with the square of CX, and the rectangle contained between bC and CX

*^{cor. 1. pr.} is * equal to the square of CA; also the
^{24 of this.} square of CX is equal ‡ to the rectangle
^{‡²⁷ of this.} contained between BT and TA; wherefore the rectangle contained between bX

and

and XC is equal to the square of CA, together with the rectangle contained between BT and TA, that is * the square of CT; consequently the rectangle contained between bH and HY, the segments of the tangent, intercepted between the point of contact and the conjugate diameters cut by it, is † equal to the square of † CG, the semi-diameter conjugate to HK, drawn through the point of contact. ^{6 E. 1. 2.} ^{14 E. 1. 5.}

Therefore, *If any tangent to an hyperbola cut two conjugate diameters, the rectangle contained between the segments of the tangent, intercepted between the point of contact and these diameters, is equal to the square described upon the semi-diameter, conjugate to that drawn through the point of contact; which was to be demonstrated.*

PROPOSITION XXXI.

THEOREM XXX.

If from any point in the periphery of an hyperbola, an ordinate be drawn to a transverse diameter, and from the vertex of the same diameter a line be drawn perpendicular to it, e-

Q q

qual

qual to its latus rectum, the square of the semi-ordinate is equal to the rectangle contained between the absciss and the latus rectum, exceeding in figure by a rectangle similar to the rectangle contained between the diameter and its latus rectum.

LET AC be any hyperbola, AB a transverse diameter, and D the center, and from C let CE be drawn an ordinate to the diameter AB, cutting it in F, and from its vertex A let AG be drawn at right angles to AB, equal to its latus rectum, and complete the rectangle BAGH; through F draw FK parallel to AG, which cuts BG produced in K; I say, the square described upon FC the semi-ordinate, is equal to the rectangle contained between the absciss AF, and latus rectum AG, exceeding in figure by a rectangle similar to BAGH, contained between the diameter and its latus rectum. For * through K draw KL parallel to AF, and complete the parallelogram GMKL;

* 31 E. 1. 1. and its latus rectum. For * through K draw KL parallel to AF, and complete the parallelogram GMKL;

† 4 E. 1. 6. Then because AG is parallel to FK, † as BA is to AG, so is BF to FK. But as BA is to AG, ‡ so is the rectangle contained

‡ cor. 7. pr. 22 of this. be.

between BF and FA to the square of CF ;
 wherefore as BF is to FK, so is the rect-
 angle contained between BF and FA to
 the square of CF ; and as BF is to FK, so
 * is the rectangle contained between BF * 1 E. 1. 6.
 and FA, to the rectangle contained be-
 tween FK and FA ; therefore † as the rect- † 11 E. 1. 5.
 angle contained between BF and FA is to
 the square of CF, so is the same rectangle
 contained between BF and FA to the rect-
 angle contained between FK and FA ; con-
 sequently the rectangle contained between
 FK and FA is ‡ equal to the square of the ‡ 9 E. 1. 5.
 semi-ordinate CF ; and the rectangle con-
 tained between FK and FA is a rectangle
 contained between the absciss AF, and the
 latus rectum AG exceeding in figure by
 the rectangle GMKL, which * is similar to * 24 E. 1. 6 ;
 the rectangle BAGH, contained between
 the diameter and its latus rectum. After
 the same manner may it be demonstrated,
 that the rectangle contained between FK
 and FA is equal to the square of CF, al-
 though AG be not at right angles to the
 diameter AB.

Therefore, *If from any point in the
 periphery of an hyperbola, an ordinate
 be drawn to any transverse diameter,*

Q q 2 and

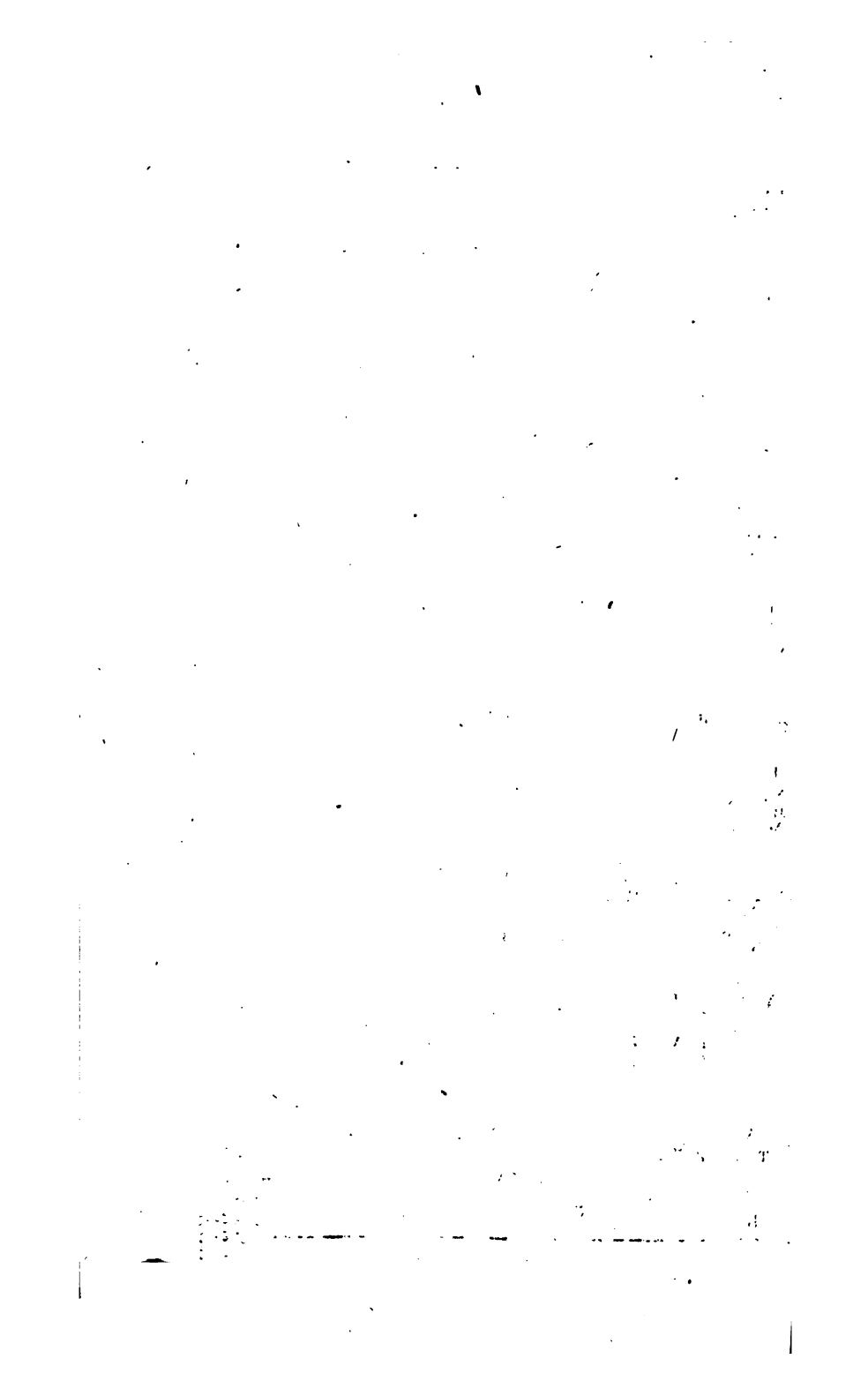
and if from the vertex of that diameter a line be drawn perpendicular to it, and equal to its latus rectum, the square described upon the semi-ordinate is equal to the rectangle contained between its absciss and the latus rectum, exceeding in figure by a rectangle similar to the rectangle contained between the diameter and its latus rectum; which was to be demonstrated (a).

PROPOSITION XXXII.

THEOREM XXXI.

If from any point in the periphery of an hyperbola an ordinate be drawn to a second diameter, as also a tangent cutting the same diameter, and from the extremity of the diameter nearest to the ordinate, a line be drawn perpendicular to the diameter, equal to its latus rectum, and the extremities of

(a) Because the squares described upon the semi-ordinates to any transverse diameter being equal to a rectangle contained between their absciss and the latus rectum, exceeding in figure by a rectangle similar to the rectangle contained between the diameter and its latus rectum, it was, that Apollonius called this curve an hyperbola.



meter and latus rectum,
 b a right line, and from
 of intersection of the dia-
 l tangent, a line be drawn
 o the line joining the extre-
 the diameter and its latus
 and cutting a line parallel
 tus rectum, drawn from the
 intersection of the diameter
 inate; the square of the se-
 ate is equal to a rectangle
 ed between the segment of the
 r, intercepted between the
 and ordinate, and the line pa-
 to the latus rectum, intercept-
 ween the diameter and the line
 r from the intersection of the
 ter and tangent, parallel to the
 oining the extremities of the di-
 er and its latus rectum.

AB be any hyperbola, CD a se-
 cond diameter, and K its center, and
 the point A let AE be drawn a se-
 ordinate to CD, cutting it in E, and
 tangent cutting it in F, and from C
 extremity of the diameter nearest to
 ordinate, let CG be drawn perpendi-
 cular

cular to, and equal to the latus rectum of CD and GD joined, and from the points E and F let EH be drawn parallel to GC, and FH parallel to GD, cutting HE in H; I say, the square of the semi-ordinate AE is equal to the rectangle contained between KE and EH. Because GC is the

** cor. 7 pr. 22 of this.* latus rectum of the diameter CD, * CD is to CG as the sum of the squares described upon KD and KE is to the square of

† cor. 1. pr. 24 of this. AE. But the square of KD † is equal to the rectangle contained between FK and KE, and the rectangle contained between

‡ 3 E. l. 2. FE and EK ‡ is equal to the rectangle contained between FK and KE, together with the square of KE; wherefore as DC is to CG, so is the rectangle contained between FE and EK to the square of AE. But because GC is parallel to HE, and GD to HF, the triangles DCG, FEH, are

** 4 E. l. 6.* equiangular, and * as DC is to CG, so is

† 11 E. l. 5. FE to HE; wherefore † as FE is to HE, so is the rectangle contained between FE

‡ 1 E. l. 6. and EK to the square of AE. But as ‡ FE is to HE, so is the rectangle contained between FE and EK to the rectangle contained between HE and EK; wherefore as the rectangle contained between FE and

EK

EK is to the square of AE, so is the same rectangle contained between FE and EK to the rectangle contained between HE and EK; consequently * the square of the ^{*9 E. 1. 3.} semi-ordinate AE is equal to the rectangle contained between HE and EK.

Therefore, *If from any point in the periphery of an hyperbola, &c. which was to be demonstrated.*

PROPOSITION XXXIII.

PROBLEM II.

Two right lines being given, mutually bisecting each other, and cutting one another at right angles; to describe two opposite hyperbolas, which have these right lines for their axes, and in such a manner as either of them may be the transverse axis.

LET AB and CD be the two given right lines, mutually bisecting each other in the point E, and cutting one another at right angles; it is required to describe two opposite hyperbolas, which have AB and CD for their two axes, so as either of them, as AB, may be the trans-

transverse one. Join BC, and from each side of the point E make EF and EG each equal to BC, and with a rule and threed, whose length is smaller than the length of the rule by AB, * describe two opposite hyperbolas LM and ON, whose foci are F and G, these hyperbolas will pass through the points A and B, and CD will be their second axis; for if one of these hyperbolas does not pass through the point B, let it, if possible, cut the line AB in any other point, as H; then because H is a point in the periphery of the hyperbola, the excess by which GH exceeds HF will be equal to the excess by which the length of the generating ruler exceeds the length of the generating threed, † that is (by the construction) AB. But because EF is equal to EG, and EA to EB, therefore AG is equal to BF; consequently the excess by which BG exceeds BF is equal to AB; therefore the excess by which GH exceeds HF is equal to the excess by which GB exceeds BF, which is absurd; therefore the hyperbola will pass through the point B. After the same manner may it be demonstrated, that the other hyperbola will pass through the point A. Also I say, that

* *def. 1 of this.*

† *1 of this.*

C and D are the extremities of their second axis; for if C is not one of its extremities, let K, if possible, be one of them, which is in the same side of the center with the point C, and join KB, then will KB be * equal to EF. But CB is equal to EF, therefore CB, KB, are equal to one another, which is absurd; wherefore the hyperbolas LM and ON are two opposite hyperbolas described, which have the two given right lines mutually bisecting each other, and cutting one another at right angles for their two axes; *which was to be done.*

PROPOSITION XXXIV,

PROBLEM III.

With a given line, and a given point out of it, to describe an hyperbola, which has the given right line for one of its axes, and which will pass through the given point.

Case I. **W**HEN the given right line is the transverse axis, in this case it is necessary that the given point be in such a position, that a line drawn from
R
K,

it, perpendicular to the given line, cut it produced.

Let AB be the given right line, and L the given point, it is required to describe an hyperbola which has AB for its transverse axis, and which will pass through the given point L; from L draw LH perpendicular to AB, cutting it produced in H; find a third proportional to BH and ^{* 11 E. 1. 6.} HL, * which let be HP, and let HP be drawn from the point H at right angles to AB; complete the right angled parallelogram BHPQ; join the line AP, which cuts the line BQ in R; find a mean proportional ^{† 13 E. 1. 6.} between AB and BR, † which let be CD, and let CD and AB be so placed as mutually to bisect each other, and cut one another at right angles; describe ^{‡ 33 of this.} ‡ the hyperbola BM, which has AB and CD for its two axes, so as AB may be the transverse one: I say, the hyperbola BM will pass through the point L. For because CD is a mean proportional between ^{* cor. prop. 20 E. 1. 6.} AB and BR, * the square of AB is to the square of CD, as AB is to BR. But AB ^{† 4 E. 1. 6.} is to BR † as AH is to HP, and as AH ^{‡ 1 E. 1. 6.} is to HP, so is ‡ the rectangle contained between AH and HB, to the rectangle con-

square of
AB as the square
of CD is to the
square of AB.

Consequently (9.11.1.)
the square of AB is
equal to the square
of AB and AB is
equal to AB.

Which is
absurd.

Therefore AB
is the transverse
axis. Q. E. I.

Describe (32. of this)
the Hyperbola BM
which has
the two
right lines AB, CD,

mutually bisecting
each other at right
angles so as AB
may be the transverse
one. I say
The Hyperbola BM
will pass through
the point L.

Which is evident
for as (above) the
square of CD is to
the square of AB as
the sum of the squares
of ED and ES is to the square
of LS. Therefore
(21. of this) the point L

is a point in the periphery
of the Hyperbola BM, which has the lines AB & CD for its
two Axes: which was to be done.

it at right angles; describe the hyperbola
BM, * which has the two right lines AB,
CD, mutually bisecting each other, and
cutting one another at right angles for its
axes, so as AB may be the transverse one;
I say, the hyperbola BM will pass through
the point L. For because AB is a mean
proportional between CD and CV, CD is
to CV † as the square of CD is to the
square of AB. But as CD is to CV, so is
TS to SX, ‡ and as TS is to SX, * so is
the rectangle contained between TS and
SE, to the rectangle contained between
SE and SX; but the rectangle contained
between SE and SX † is equal to the
square of LS; wherefore ‡ as the square
of CD is to the square of AB, so is the
rectangle contained between TS and SE
to the square of LS. But the rectangle
contained between TS and SE, * is equal
to the rectangle contained between TE
and ES, that is, the square of ED, † to-
gether with the square of SE; therefore
as the square of CD is to the square of
AB, so is the sum of the squares described
upon ED and ES to the square of LS;
consequently the point L ‡ is in the peri-
phery of the hyperbola BM; wherefore
the point L is a point in the periphery
of the Hyperbola BM, which has the lines AB & CD for its
two Axes: which was to be done.

the hyperbola BM is described, with the given line CD for its second axis, and passes through the given point L; *which was to be done.*

PROPOSITION XXXV.

PROBLEM IV.

An hyperbola being given, to find its diameters, its center, its axes, its foci and asymptotes.

LET ABC be an hyperbola, it is required to find its diameters, its center, axes, asymptotes and foci. Draw any two lines, as AC and DE, parallel to one another, and terminated both ways by the periphery in the points A, C, D and E; bisect the lines AC, DE, * in F * ^{10 E.L. I.} and G; join the line FG: Then because FG bisects two lines terminated by the hyperbola, and parallel to one another, † it is a diameter. After the same manner may any other diameter be found; and the point where they cut one another, as in H, ‡ is the center. The center H being found, as above, to find the axes: ^{† cor. 6 pr. 22 of this.} Take any point in the periphery, as A, ^{† def. 4 of this.} join

join HA, and about the center H, with the distance HA, describe a circle; if that circle cut the periphery of the hyperbola in no other point than A, then will HA be the smallest of all the transverse diameters, and consequently * is the transverse axis; but if the circle cut the hyperbola in any other point, as C, join the line AC, bisect it in F, and join HF, which cuts the hyperbola in B, HB will be half the transverse axis. For through B draw BK parallel to AF; then because AF is equal to FC, AC is † an ordinate to the diameter HB; wherefore BK, which is ‡ of this. parallel to AF, ‡ is a tangent to the hyperbola, and the angle HBK is equal to the angle AFH; but AFH * is a right angle, wherefore the angle HBK is a right angle; consequently † the diameter HB is the transverse axis. Again, with the line ‡ 45 E. l. 1. BF ‡ construct the right angled parallelogram BFLK, equal to the square of AF, and make HM equal to HB; join the line * 13 E. l. 6. ML, cutting BK in N; find * a mean proportional between MB and BN, which let be OP; let OP be bisected in the center, and cutting MB at right angles, OP will be the second axis. Because the rectangle BL

BL

BL is equal to the square of AF, wherefore * the rectangle contained between MF and FB is to the square of AF, as the rectangle contained between MF and FB is to the rectangle BL. But † as the rectangle † † E. I. 6; contained between MF and FB is to the rectangle BL, so is MF to FL, and as MF is to FL, so is MB to BN; wherefore † as † † E. I. 5; the rectangle contained between MF and FB is to the square of AF, so is MB to BN. But because OP is a mean proportional between MB and BN, MB is to BN † as the square of MB is to the square of OP; wherefore as the square of MB is to the square of OP, so is the rectangle contained between MF and FB to the square of AF; consequently OP is the second axis; and make HQ and HR each equal to OB, then will * Q and R † be the foci; also make BS and BT equal to OH and HP, join the lines HT and HS, these lines HT and HS are † the asymptotes. † def. 12 of this. Hence the diameters, the center, the axes, the foci and asymptotes of the given hyperbola ABC are found; *which was to be done.*

PROPOSITION XXXVI.

PROBLEM V.

Two right lines being given cutting one another, and a point lying between them, to describe an hyperbola which will have the given right lines for its asymptotes, and to pass through the given point.

LET AB and BC be two right lines cutting one another in B, and D any point lying between AB and BC, it is required to describe an hyperbola which will pass through the point D, and which has the given lines AB and BC for its asymptotes. Bisect the angle ABC * with the line BE, and from D draw DF parallel to BE, cutting AB in F, and BC produce in G, and † make the squares of BE and BH each equal to the rectangle contained between GD and DF; from the point E draw the line EA perpendicular to BE, cutting AB and BC in the points A and ‡ 31 E. I. 1. C, and through B ‡ draw KL parallel to AC, and make KB and BL each equal to AE and EC, and describe the hyperbola * 33 of this. EM, * which has the two right lines HE, KL,

KL, mutually bisecting each other, and cutting one another at right angles for its axes, so as HE may be the transverse one; I say, the hyperbola EM has the given right lines AB and BC for its asymptotes, and passes through the point D. For because AEC is drawn from the vertex of the transverse axis at right angles to it, therefore AEC is * a tangent to the hyperbola; and since AC is equal to the second axis KL, wherefore AB and BC are † the asymptotes. Again, because the line GFD is drawn parallel to the transverse axis, from a point within the angle ABC, contained between the asymptotes, and the rectangle contained between GD and DF, the segments of the line intercepted between the given point and each asymptote, is equal to the square of BE half the transverse axis, the point D is ‡ in the periphery of the hyperbola; wherefore the hyperbola EM is described, which has the given lines AB, BC, for its asymptotes, and passes through the given point D; which was to be done.

* cor. 3. pr.
2 of this.

* def. 12 of
this.

‡ converse
part 2. pr.
9 of this.

S f

PRO

PROPOSITION XXXVII.

PROBLEM VI.

Two right lines being given, mutually bisecting each other, to describe two opposite hyperbolas, which have either of the given lines a transverse diameter, and the other its second conjugate diameter.

LET AB, CD, be two right lines, mutually bisecting each other in the point E, it is required to describe two opposite hyperbolas, which have one of these lines, as AB, for a transverse diameter, and the other, CD, its second conjugate. From
 * 31 E. l. 1. B draw FBG parallel * and equal to CD; so as FG is bisected in B; join FE and
 † 36 of this EG; then † describe the hyperbola HBK, which has FE and EG for its asymptotes, and which passes through the point B lying between them, the line BA is a transverse diameter, and CD is its second conjugate. For because FG, terminated by the asymptotes; is bisected in B, the point where it meets with the hyperbola, wherefore FG is ‡ a tangent; and because CD is drawn through the center parallel and equal

‡ conv. def.
 13 of this.

equal to FG, and bisected in it, wherefore CD is * the second diameter to the transverse AB, drawn through the point of contact; wherefore HBK is an hyperbola described, which has one of the given right lines, mutually bisecting each other, for a transverse diameter, and the other for its second conjugate; *which was to be done.* After the same manner, if the asymptotes FE, GE, be produced, may the opposite hyperbola be described passing through the point A.

PROPOSITION XXXVIII.

PROBLEM VII.

With a given right line, and a given point out of it, from which point a line is drawn cutting the given line, it is required to describe an hyperbola which has the given line for a diameter, and which will pass through the given point, so as the other line drawn from the given point may be a semi-ordinate to that diameter.

Case I. **W**HEN the given line is a transverse diameter, in this case
S f 2 it

it is necessary that the given point be in such a position, that the line drawn from it, which is to be a semi-ordinate to the given diameter, cut it produced.

Let AB be the given right line, and C a given point out of it, and from C let CD be drawn cutting AB produced in D, it is required to describe an hyperbola, which has AB for a transverse diameter, passing through the point C, so as CD may be a semi-ordinate to AB; find a
 * 11 E. 1. 6. third proportional to DB and CD, * as DE, and let DE be drawn perpendicular to AB; complete the right angled parallelogram BDEF; join the line AE, which
 † 13 E. 1. 6. cuts BF in G; find † a mean proportional between AB and BG, which let be HK; let HK be placed so as to be parallel to CD, and to bisect AB and itself in the
 ‡ 13 of this. point L; and ‡ describe the hyperbola BR, which has AB and HK for two conjugate diameters, having AB for the transverse one: I say, the hyperbola BR will pass through the given point C, and CD will be a semi-^{ordinate} to AB. For because HK is a mean proportional between
 * cor. 20 E. 1. 6. AB and BG, * as AB is to BG, so is the square of AB to the square of HK. But
 as

as AB is to BG, * so is AD to DE, and * 4E. 1. 6.
as AD is to DE, so is the rectangle contained between AD and DB, to the rectangle contained between DE and DB;
wherefore † as the rectangle contained between AD and DB is to the rectangle contained between DE and DB, so is the square of AB to the square of HK; but the rectangle contained between DE and DB is equal to the square of CD; wherefore as the square of AB is to the square of HK, so is the rectangle contained between AD and DB to the square of CD; consequently ‡ the point C is in the periphery of the hyperbola BR, and CD is a semi-ordinate to the diameter AB; therefore BR is an hyperbola described, which has the given line AB for a transverse diameter, and passes through the given point C, having CD for a semi-ordinate to that diameter; *which was to be done.*

Case 2. When the given line is a second diameter.

Let HK be the given right line, and C the given point out of it, and from C let CM be drawn cutting HK in M; it is required to describe an hyperbola which has the given line HK for a second diameter, and

326 *Elements of Conic Sections. Book III.*

and which will pass through the given point C , so as CM may be a semi-ordinate to HK . Bisect HK in L ; find a third ^{*11 E. I. 6.} proportional $*$ to ML and LK , which let be LN ; draw MO perpendicular to HK , and let MO be a third proportional to ML and MC ; join ON , and from H draw HQ ^{†31 E. I. 1.} parallel to MO , and from K [‡] draw KQ parallel to ON , cutting HQ in Q ; find ^{‡13 E. I. 6.} $‡$ a mean proportional between HQ and HK , which let be AB , and let it cut HK in L , so as to be bisected and parallel ^{*33 of this.} to CM ; describe the hyperbola $*$ which has AB and HK mutually bisecting each other for two conjugate diameters, so as AB may be the transverse diameter, which let be BR : I say, the point C is in the periphery of the hyperbola BR . For because AB is a mean proportional between HK and KQ , ^{† cor. 20 E. I. 6.} $† HK$ is to HQ as the square of HK is to the square of AB ; and because HQ is parallel to MO , and QK to ON , the triangles QHK and OMN are ^{‡ 4 E. I. 6.} equiangular, and $‡$ as KH is to HQ , so ^{* 1 E. I. 6.} $*$ is NM to MO ; but $*$ as NM is to MO , so is the rectangle contained between MN and ML , to the rectangle contained between MO and ML , that is, the square of

of CM; wherefore * as the square of HK ^{*11E.1.3.} is to the square of AB, so is the rectangle contained between MN and ML to the square of CM; but because LK or LH is a mean proportional between ML and LN, the rectangle contained between LM and LN † is equal to the square of LK; and † ^{17E.1.6.} the rectangle contained between LM and MN ‡ is equal to the rectangle contained ‡ ^{3.E.1.2.} between LM and LN, that is, the square of LK, together with the square of ML; wherefore as the square of HK is to the square of AB, so is the sum of the squares described upon LK and LM to the square of CM; consequently the point C * is in ^{*22 of this} the periphery of the hyperbola BR; wherefore BR is an hyperbola described, which has the given line HK for a second diameter, passing through the given point C, having CM for a semi-ordinate to that diameter; *which was to be done.*

PROPOSITION XXXIX.

THEOREM XXXII.

If a cone be cut by a plane through its axis, also by another plane cutting the base of the cone in the direction of
of.

of a right line, which is perpendicular to the base of the triangle made by the plane passing through the axis; and if the common section of the cutting plane, and triangle made by the plane through the axis, and one side of the same triangle, meet in a point without the vertex of the cone; the figure made by the common section of the cutting plane and conical surface is an hyperbola, which has the common section of the cutting plane, and the triangle made by the plane through the axis, for a transverse diameter.

LET A be the vertex, and BDC the circular base of a cone, which is cut by a plane through its axis, and let ABC be the triangle made by that section, and let it be cut by another plane cutting the base of the cone in the direction of the line DF, which is perpendicular to the base BC of the triangle ABC, and let EG, the common section of the cutting plane and triangle meet AC, one side of that triangle ABC produced in H, which is without the vertex of the cone, and let
DEF

DEF be the figure made by the cutting plane and conical surface: I say, the figure DEF is an hyperbola, which has EG the common section of the cutting plane, and triangle through the axis, for a transverse diameter. For take any point, as K, in the section DEF, and from K draw KL parallel to DF, cutting EG in L, and through L draw * MN parallel to BC. ^{* 31 E. I. ¶}

Then because KL is parallel to DG, and ML to BG, the plane which passes through ML and LK † is parallel to the plane which ^{† 15 E. I. ¶} passes through BG and GD, that is, to the base of the cone BDC; wherefore ‡ the ^{‡ 27 l. 1. of this.} plane passing through ML and LK is a circle which has MN for a diameter; and because BC is perpendicular to DF, wherefore * MN is perpendicular to KL; there- ^{* 10 E. I. ¶} fore † the square of KL is equal to the ^{† 35 E. I. ¶} rectangle contained between ML and LN; also the square of DG is equal to the rectangle contained between BG and GC. Therefore the square of DG is to the square of KL, as the rectangle contained between BG and GC, is to the rectangle contained between ML and LN; but the rectangle contained between BG and GC is to the rectangle contained between ML and LN

T r

* in

330 *Elements of Conic Sections.* Book III.

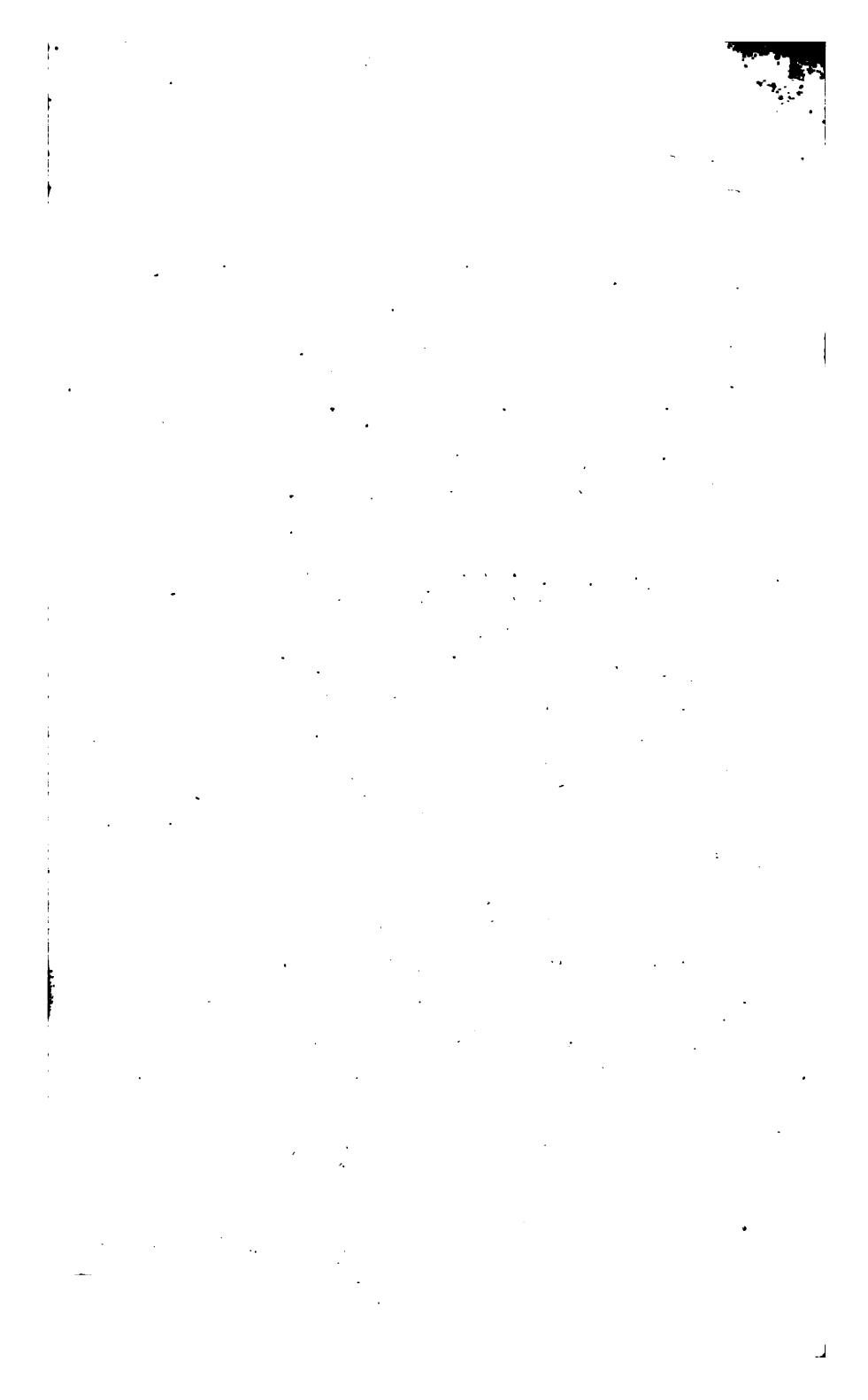
*_{23 E. l. 6.} * in a ratio compounded of BG to ML,
†_{4 E. l. 6.} and of GC to LN; but BG is to ML † as
GE is to LE, and GC is to LN as GH is
to LH; wherefore the ratio compounded
of BG to ML, and of GC to LN, is e-
qual to the ratio compounded of GE to
LE, and of GH to LH; therefore the
square of DG is to the square of KL, in a
ratio compounded of GE to LE, and of
†_{23 E. l. 6.} GH to LH. But the rectangle contained
between GE and GH, is to the rectangle
contained between LE and LH, in a ratio
compounded of GE to LE, and of GH
*_{11 E. l. 5.} to LH; wherefore * the square of DG is
to the square of KL, as the rectangle con-
tained between GE and GH, is to the
rectangle contained between LE and LH.
†_{38 of this.} Let an hyperbola be described, † which
has HE for a transverse diameter, and DG
for a semi-ordinate to that diameter; then
because KL is drawn parallel to the semi-
ordinate DG, and the square of DG is to
the square of KL, as the rectangle con-
tained between GH and GE, is to the
rectangle contained between LE and LH;
†_{68, 4 pr.} wherefore the point K is in ‡ the periphe-
‡_{22 of this.} ry of the same hyperbola. After the same
manner may it be demonstrated, that e-
very

very point of the section DEF is in the periphery of the same hyperbola; consequently the section DEF is an hyperbola, which has EG the common section of the cutting plane, and triangle made by the plane through the axis, for a transverse diameter.

Therefore, *If a cone be cut by a plane through its axis, also by another plane cutting the base of the cone in the direction of a right line, which is perpendicular to the base of the triangle made by the plane passing through its axis; and if the common section of the cutting plane, and triangle made by the plane through the axis, and one side of the same triangle produced, meet in a point without the vertex of the cone; the figure made by the common section of the cutting plane and conical surface is an hyperbola, which has the common section of the cutting plane, and the triangle made by the plane through the axis, for a transverse diameter; which was to be demonstrated.*

F I N I S.

JUL 12 1920



K
L

C